Math 650 Exam 2 November 1, 2018

Solution Key

1. Consider the system

\[ \dot{x} = x + x^2y, \quad \dot{y} = x^2 - x - y. \]

(a) Show that the origin is an equilibrium and is a saddle.

(b) Approximate the unstable manifold

\[ W^u(0, 0) = \{ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \} \]

by finding the first three nonzero terms in the above approximation.

(c) Can you guess what the stable manifold \( W^s(0, 0) \) is?

Sln Key: (a) It’s clear that the origin is an equilibrium. The coefficient matrix of the linearization at \((0, 0)\) is \( A = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \), which has e-values \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \) with associated e-vectors \( V_1 = (1, -1/2)^T \) and \( V_2 = (0, 1)^T \).

(b) The unstable manifold passes through the origin and is associated to \( \lambda_1 \) and \( V_1 \), and hence, \( a_0 = 0 \) and \( a_1 = -1/2 \). Thus, it is the graph of \( y = -1/2x + a_2 x^2 + a_3 x^3 + \cdots \). One has, on one hand, \( \dot{y} = (-1/2 + 2a_2 x + 3a_3 x^2 + \cdots) \dot{x} \); that is,

\[ \dot{y} = (-1/2 + 2a_2 x + 3a_3 x^2 + \cdots) (x - 1/2x^3 + a_2 x^4 + a_3 x^5 + \cdots). \]

On the other hand, from the equation for \( y \), one gets

\[ \dot{y} = x^2 - x + 1/2x - a_2 x^2 - a_3 x^3 + \cdots. \]

Comparing the terms of like powers in \( x \), one gets \( a_2 = 1/3 \) and \( a_3 = -1/16 \).

Thus, the unstable manifold is the graph of

\[ y = -\frac{1}{2}x + \frac{1}{3} x^2 - \frac{1}{16} x^3 + \cdots. \]

(c) The stable manifold is the \( y \)-axis since the \( y \)-axis is invariant and is tangent to \( V_2 \) (associated to \( \lambda_2 = -1 \)).

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2. (a) Show that the origin is a non-hyperbolic equilibrium of

\[ \dot{x} = x^2 y + y^2, \quad \dot{y} = -y + x^2 - xy. \]

(b) Sketch the phase plane portrait of the linearized system at (0, 0).

(c) Approximate a center manifold \( W^c(0, 0) \) and sketch the phase plane portrait of the nonlinear system near the origin.

**Soln Key:** (a) The coefficient matrix of the linearization at (0, 0) is \( A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \), which has e-values \( \lambda_1 = 0 \) and \( \lambda_2 = -1 \) with associated e-vectors given by the \( x \)-axis and the \( y \)-axis, respectively. In particular, the origin is a non-hyperbolic equilibrium.

(b) For linearized system, the \( x \)-axis consists of equilibria, each vertical line is invariant and the corresponding equilibrium is stable within the line.

(c) A center manifold \( W^c(O) \) is given by \( y = a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = x^2(a_2 + a_3 x + a_4 x^2 + \cdots) \). One has, on one hand,

\[ \dot{y} = (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) \dot{x} = x^5(2a_2 + 3a_3 x + \cdots)(a_2 + a_3^2 + (a_3 + 2a_2a_3)x + \cdots); \]

on the other hand, \( \dot{y} = -y + x^2 - xy = (1 - a_2)x^2 - (a_2 + a_3)x^3 - (a_3 + a_4)x^4 + \cdots. \)

Comparing the terms of like powers in \( x \), one gets \( a_2 = 1, a_3 = -1, a_4 = 1, \) etc..

Thus, a center manifold \( W^c(O) \) is given by \( y = x^2 - x^3 + x^4 + \cdots. \)

On \( W^c(O), \dot{x} = x^4(2 - 3x + \cdots) \), and hence, the equilibrium is a saddle-node.
3. (a) Show that the following system is a Hamiltonian.

\[ \dot{x} = x + 2y, \quad \dot{y} = -4x^3 - y. \]

(b) Find a Hamiltonian function \( H(x, y) \) for the system.

(c) Find the equilibria and determine their types.

**Soln Key:** (a) \( f_y + g_y = (x+2y) + (-4x^3-y) = 1-1 = 0 \). So the system is a Hamiltonian.

(b) It follows from \( h_y = f = x + 2y \) and \( H_x = -g = 4x^3 + y \) that \( H(x, y) = x^4 + xy + y^2 \).

(c) Equilibria of the dynamical system are critical points of \( H \) and they are

\[ O = (0, 0), \quad E_+ = \left(-\frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}\right), \quad E_- = \left(-\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right). \]

It follows from \( H_{xx} = 12x^2, \ H_{yy} = 2, \ H_{xy} = 1 \) that \( D(x, y) = H_{xx}H_{yy} - H_{xy}^2 = 24x^2 - 1 \).

Thus, \( D(O) = -1 < 0 \) so that \((0, 0)\) is a saddle critical point of \( H \), and hence, a saddle for the dynamical system.

\( D(E_{\pm}) = 3 > 0 \) and \( H_{yy}(E_{\pm}) > 0 \) so \( E_{\pm} \) are local mini. of \( H \), and hence, are centers for the dynamical system.

**Remark.** An alternative way to determine the types of the equilibria is to determine the eigenvalues of the linearization at each equilibria. But, for the equilibria \( E_{\pm} \) where eigenvalues are pure imaginary, one needs the fact that the system is a Hamiltonian to claim that they are centers for the nonlinear dynamical system.
4. (a) Find a potential function $V(x)$ for the Newtonian system $\ddot{x} = 3x^2$.

(b) Sketch the phase plane portrait for its equivalent form $\dot{x} = y$, $\dot{y} = 3x^2$.

**Soln Key:** (a) A potential function $V(x)$ is given by

$$V(x) = -\int 3x^2 \, dx = -x^3.$$  

It has only one critical point $x = 0$, which is a saddle critical point of $V$. Therefore, the origin is a cusp equilibrium of the equivalent planar system.

(b) The phase plane portrait is sketched below.
5. Sketch the phase plane portrait for the Newtonian system

\[ \dot{x} = y, \quad \dot{y} = -V'(x) \]

where the graph of \( V \) is given below. Note that \( V(-1) = V(1) \).
6. Sketch the phase plane portrait for the Newtonian system

\[ \dot{x} = y, \quad \dot{y} = -V'(x) \]

where the graph of \( V \) is given below. Note that \( V(-1) < V(1) \).