Steady states for shear flows of a liquid-crystal model: multiplicity, stability, and hysteresis

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Abstract

In this work, we study shear flows of a fluid layer between two solid blocks via a liquid-crystal type model proposed in [*C. H. A. Cheng, et al., A liquid-crystal model for friction, PNAS* **21** *(2007), 1-5*] for an understanding of frictions. A characterization on the existence and multiplicity of steady-states is provided. Stability issue of the steady-states is examined mainly focusing on bifurcations of zero eigenvalues. The stability result suggests that this simple model exhibits hysteresis, and it is supported by a numerical simulation.

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1 Introduction

In this work, we study a liquid-crystal type model for friction of a fluid layer between two solid blocks, particularly, for the case of geologic fault, proposed in [3]. The model is motivated by the Ericksen-Leslie continuum theory for nematic liquid-crystals ([7, 10]). The state variables of nematic liquid-crystals are the velocity field together with a director field. The latter is an attempt to take into consideration of the micro-geometry of the molecules forming the material; in particular, for nematic liquid-crystals, they are treated as rod-like particles. The model in [3] is a simplified version for liquid-crystals but, at very basic level, mimics the nematic liquid-crystals continuum system of Ericksen-Leslie.

We begin with a brief account of the model and refer the readers to [3] for more details and [7, 10, 2, 4] for the continuum theory of "real" nematic liquid-crystal formulation. Consider a fluid layer of prescribed thickness between two solid blocks with the blocks sliding in opposite directions at a prescribed relative slip velocity. The material of the fluid layer will be treated as rod-like liquid-crystals, in particular, the state of a material element is determined by its spatial location and the direction of the rod. The following continuum model was proposed in [3] (see the reference for a derivation and discussions).

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathrm{div}(v(\mathbf{r})\nabla\mathbf{u}) - \frac{1}{\rho}\nabla p, \quad \text{in } \Omega \tag{1.1}$$

$$\mathbf{r}_t + \nabla\mathbf{r}\cdot\mathbf{u} - \nabla\mathbf{u}\cdot\mathbf{r} = \delta\Delta\mathbf{r}, \quad \text{in } \Omega \tag{1.2}$$

where $\mathbf{u}$ is the velocity of the fluid, $\mathbf{r}$ is the director field, ρ is the density, $v(\mathbf{r})$ is the kinematic viscosity, and p is the pressure, δ is the relaxation parameter, and Ω denotes the region bounded by the two solid blocks.

The domain Ω is taken as $\Omega = (-\infty,\infty)\times(0,L)$ with coordinates (x,y). Without loss of generality, we take $L = 1$. Assume the horizontal pressure gradient to be zero so that the flow is driven by the imposed slip velocity $\bar{\mathbf{u}}$ of the upper boundary of the channel while the lower boundary can be set still. Let $\mathbf{u} = (u_1(x,y), u_2(x,y))$ and $\mathbf{r} = (r_1(x,y), r_2(x,y))$. We recall further simplifications assumed in [3]. It is assumed that

$$u_1(x,y) = u(y),\ u_2 = 0; \quad r_1(x,y) = r(y),\ r_2 = 1.$$

Under this assumption and with $v(r) = v(\mathbf{r})$, one obtains a one-dimensional version of the model:

$$u_t = (v(r)u_y)_y, \quad r_t = \delta r_{yy} + u_y, \quad \text{for } y\in(0,1), \tag{1.3}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \bar{u}; \quad r(0,t) = r(1,t) = 0. \tag{1.4}$$

The kinematic viscosity $v(r)$ is assumed to depend on the director field r via a model

$$v(r) = \alpha(\theta)v_1 + (1 - \alpha(\theta))v_0$$

for some decreasing function α with $\alpha(0) = 1$ and $\alpha(\pi/2) = 0$ and $0 < v_0 < v_1$, where θ is the angle of $\mathbf{r}$ from the vertical. The function $\alpha(\theta)$ determines the type of frictions modeled. In [3], the authors introduced the above model (1.1) and (1.2), and used numerical simulations to examine the behavior of solutions of (1.3) and (1.4) that allow them to compare with the empirical rate-and-state law.

In this paper, we consider a general $v(r)$ and assume

$$\begin{gathered} 0 < v_0 \le v(r) \le v_1, \quad v'(r) \le 0 \text{ and} \\ \text{either } v'(r) = 0 \text{ for large } r \text{ or } \lim_{r\to\infty} \frac{(v'(r))^2}{v''(r)} = \mu_0. \end{gathered} \tag{1.5}$$

We remark that the existence of the limit in the assumption (1.5) implies implicitly that $v''(r) \neq 0$ for large r and, it turns out $\mu_0 = 0$ (see Lemma 2.3). We will then first examine steady-state solutions of (1.3) and (1.4). This is rather easy and we are able to give a complete characterization on the existence and multiplicity of steady-states. Stability issue of the steady-states is then examined, particularly, for cases where multiple steady-states exist. We identify conditions on $\bar{u}$ so that zero is an eigenvalue for the linearization of a steady-state associated to $\bar{u}$ and study the bifurcation of the zero eigenvalue for nearby $\bar{u}$. Quite interestingly, our stability result suggests that this simple model possesses hysteresis; more precisely, when one applies dynamic boundary conditions $\bar{u}(t)$ in two manners, one with slowly increasing $\bar{u}(t)$ from zero to large, and the other in the reversal way, the solution of (1.3) and (1.4) for the second setup is *not* the reverse of the first.

The paper is organized as follows. A characterization (Theorem 2.2) on the existence and multiplicity of steady-state solutions of (1.3) and (1.4) is provided in Section 2 followed by an example of $v(r)$ for which multiple steady-states exist. In Section 3, we apply the energy estimate to establish the L^2 linear stability of steady-states with small $\bar{u}$ (Theorems 3.1). Section 4 is devoted to the study of zero-eigenvalue and its bifurcation of steady-state solutions. An explicit condition (Theorem 4.6) on steady-states that possess a zero eigenvalue and a key formula (Proposition 4.9) that determines the bifurcation of the zero eigenvalue are given. Section 5 is devoted to the derivation of the formula. The stability result suggests a mechanism for a hysteresis phenomenon for this model problem. A numerics is presented in Section 6 to support and illustrate the expected hysteresis phenomenon.

2 Existence and multiplicity of steady-states

In this section, we will characterize the existence and multiplicity of steady-state solutions of (1.3) and (1.4) under the assumption (1.5) on the kinematic viscosity v.

For definiteness, we assume $\bar{u} > 0$ in (1.4). The steady-state problem of (1.3) and (1.4) is, for some positive constant M^2 (see (2.7)),

$$\begin{aligned} v(r)u' = & M^2, \quad \delta r'' + u' = 0, \quad y \in (0,1) \\ u(0) = 0, \quad & u(1) = \bar{u} > 0, \quad r(0) = r(1) = 0, \end{aligned} \tag{2.6}$$

where prime denotes the derivative with respect to y.

It follows that

$$u(y) = u(0) + M^2 \int_0^y \frac{1}{v(r(s))} ds.$$

In view of the boundary conditions $u(0) = 0$ and $u(1) = \bar{u} > 0$, we have

$$M^2 = \bar{u} \left\{ \int_0^1 \frac{1}{v(r(y))} dy \right\}^{-1} > 0. \tag{2.7}$$

Set

$$f(r) = \int_0^r \frac{1}{v(\tau)} d\tau. \tag{2.8}$$

It is clear that f is strictly increasing. Let g be the inverse function of f. The following lemma is a simple consequence of (1.5).

Lemma 2.1. *$f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, $f'(r) = 1/v(r) > 0$, $f''(r) = -v'(r)/v^2(r) \geq 0$; $g'(s) = v(g(s)) \in [v_0, v_1]$, $g''(s) \leq 0$.*

Define a function

$$D(\beta) = \beta \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt \quad \text{for} \quad \beta > 0. \tag{2.9}$$

The existence and multiplicity result for steady-states is

Theorem 2.2. *For any $\bar{u} > 0$, the set of solutions of the boundary value problem (2.6) is in one-to-one correspondence with the set of solutions β of $\bar{u} = 4\delta D(\beta)$. In particular, there always exists at least one solution.*

Proof. The steady-state problem (2.6) reduces to

$$r'' + \frac{M^2}{\delta v(r)} = 0, \quad r(0) = r(1) = 0, \tag{2.10}$$

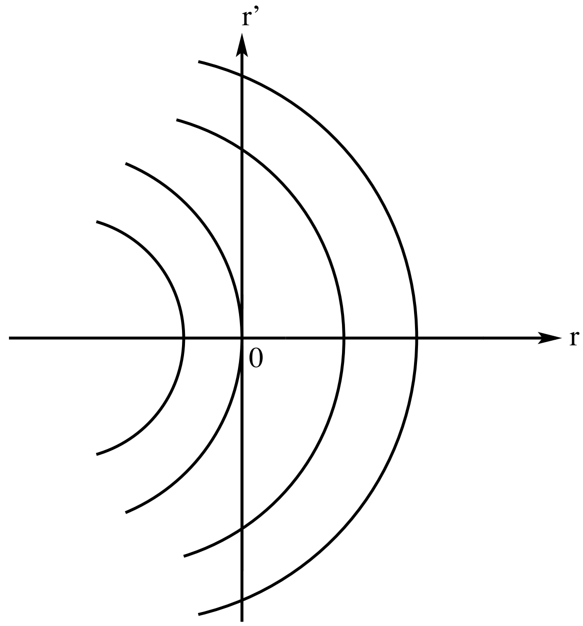

Figure 1: *The phase portrait of system (2.10) in the* (r, r')*-plane.*

subject to the condition (2.7). For any fixed $M^2 > 0$, equation (2.10) is a Newtonian with the potential $M^2 f(r)/\delta$ or with the Hamiltonian

$$H(r, r') = \frac{1}{2}(r')^2 + \frac{M^2}{\delta} f(r), \tag{2.11}$$

where $f(r)$ is given in (2.8). The phase portrait is sketched in Figure 1.

We claim that, if $r(y)$ is a solution of the boundary value problem (2.10), then $r(y) \geq 0$ for $y \in (0,1)$, $r(y)$ is symmetric about $y = 1/2$ and $r'(1/2) = 0$. In fact, it follows from the equation in (2.10) that $r(y)$ is strictly concave downward. The boundary condition $r(0) = r(1) = 0$ then implies that $r(y) > 0$ for $y \in (0,1)$ and there is a unique $y^* \in (0,1)$ so that $r'(y^*) = 0$. Set $r(y^*) = \alpha$ and $r_1(y) = r(2y^* - y)$. Then $r_1(y)$ satisfy the second-order equation in (2.10) and the initial conditions $r_1(y^*) = r(y^*) = \alpha$ and $r_1'(y^*) = -r'(y^*) = 0$. By uniqueness of initial value problems, we have $r(y) = r_1(y)$; in particular, $r_1(1) = r(2y^* - 1) = r(1) = 0$. Since $r(y) = 0$ implies $y = 0$ or $y = 1$, we have either $2y^* - 1 = 1$ or $2y^* - 1 = 0$; that is, either $y^* = 1$ or $y^* = 1/2$. We thus conclude $y^* = 1/2$ since $y^* \in (0,1)$, and hence, $r(y) = r_1(y) = r(1-y)$.

It now follows from (2.11), $r(1/2) = \alpha$ and $r'(1/2) = 0$ that

$$\frac{1}{2}(r')^2 + \frac{M^2}{\delta} f(r) = \frac{M^2}{\delta} f(\alpha),$$

and hence, for $y \in (0, 1/2)$, $r'(y) \geq 0$ and

$$r' = \sqrt{\frac{2}{\delta}} M \sqrt{f(\alpha) - f(r)} \quad \text{or} \quad M dy = \sqrt{\frac{\delta}{2}} \frac{dr}{\sqrt{f(\alpha) - f(r)}}. \tag{2.12}$$

Integrate from $y = 0$ to $y = 1/2$ to get

$$M = \sqrt{2\delta} \int_0^\alpha \frac{dr}{\sqrt{f(\alpha) - f(r)}}.$$

Note that $r'(0) = \sqrt{2\delta^{-1}} M \sqrt{f(\alpha)}$ and

$$\begin{aligned}\int_0^1 \frac{dy}{v(r(y))} =& 2\int_0^{1/2} \frac{dy}{v(r(y))} = -\frac{2\delta}{M^2} \int_0^{1/2} r''(y)dy \\ =& -\frac{2\delta}{M^2} \left(r'(1/2) - r'(0)\right) = \frac{2\delta}{M^2} r'(0) = \frac{2\sqrt{2\delta}}{M} \sqrt{f(\alpha)}.\end{aligned}$$

The relation (2.7) then imposes that

$$\bar{u} = 2M\sqrt{2\delta}\sqrt{f(\alpha)} = 4\delta\sqrt{f(\alpha)} \int_0^\alpha \frac{dr}{\sqrt{f(\alpha) - f(r)}}.$$

Let $\beta = f(\alpha)$ or equivalently $\alpha = g(\beta)$. In terms of β, we have,

$$M = \sqrt{2\delta} \int_0^\beta \frac{g'(s)}{\sqrt{\beta - s}} ds = \sqrt{2\delta}\beta^{1/2} \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt, \tag{2.13}$$

$$\bar{u} = 4\delta\beta^{1/2} \int_0^\beta \frac{g'(s)}{\sqrt{\beta - s}} ds = 4\delta\beta \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt = 4\delta D(\beta). \tag{2.14}$$

It follows that, given any $\bar{u} > 0$, if $\beta > 0$ is a solution of (2.14), then there is a steady-state solution. It is also clear from the construction of the steady-state solution and the monotonicity of $f(r)$ that different β values provide different steady-state solutions. Therefore, the set of steady-states is in one-to-one correspondence with the set of solutions β of equation (2.14).

Since $0 < v_0 \le g'(s) \le v_1$ from Lemma 2.1, one has $D(\beta) \to 0$ as $\beta \to 0$ and $D(\beta) \to \infty$ as $\beta \to \infty$. Thus, for any $\bar{u} > 0$, there exists at least one $\beta > 0$ such that (2.14) is satisfied. This completes the proof. $\square$

Next, we provide a condition on $\bar{u}$ so that the corresponding boundary value problem (2.6) has a unique solution and an example of $v(r)$ for which the boundary value problem (2.6) has multiple solutions for a range of $\bar{u}$.

Lemma 2.3. *Assumption (1.5) implies $\mu_0 = 0$.*

Proof. Assume, on the contrary, that $\mu_0 \neq 0$. If $\mu_0 < 0$, then the existence of $\lim_{r\to\infty} \frac{(v'(r))^2}{v''(r)} = \mu_0$ implicitly implies that, for some large r_0, $v''(r) \le 0$ if $r \ge r_0$ and $v'(r_0) < 0$. It follows that $v'(r) \le v'(r_0) < 0$, and hence, $v(r) \le v(r_0) + (r - r_0)v'(r_0) \to -\infty$ as $r \to \infty$ that contradicts to $v(r) \ge v_0 > 0$. Therefore, $\mu_0 > 0$. Denote, for $r > r_0$, $\frac{(v'(r))^2}{v''(r)} = \rho(r)$. Then $\rho(r) \to \mu_0$ as

$r \to \infty$. Assume $\rho(r) \geq \mu_0/2$ for $r \geq r_*$ for some $r_* > r_0$. Solve the equation $\rho(r)(v')' = (v')^2$ to get, for $r \geq r_*$,

$$v'(r) = -\frac{1}{\int_{r_*}^{r} \frac{1}{\rho(\tau)} d\tau - \frac{1}{v'(r_*)}}.$$

Hence,

$$v(r) = v(r_*) - \int_{r_*}^{r} \frac{1}{\int_{r_*}^{s} \frac{1}{\rho(\tau)} d\tau - \frac{1}{v'(r_*)}} ds.$$

It follows from, for $r \geq r_*$, $\rho(r) \geq \mu_0/2$ that

$$\int_{r_*}^{r} \frac{1}{\rho(\tau)} d\tau \leq \frac{2}{\mu_0}(r - r_*).$$

Therefore,

$$v(r) \leq v(r_*) - \int_{r_*}^{r} \frac{1}{\frac{2}{\mu_0}(s - r_*) - \frac{1}{v'(r_*)}} ds \to -\infty \ \text{ as } \ r \to \infty.$$

The latter contradicts to $v_0 \leq v(r) \leq v_1$. We thus conclude $\mu_0 = 0$. $\square$

Corollary 2.4. *Assume (1.5). There exist* $0 < \beta_1 < \beta_2$ *such that* $D'(\beta) > 0$ *for* $0 < \beta < \beta_1$ *and for* $\beta > \beta_2$. *Hence, for* $\bar{u} \in (0, 4\delta D(\beta_1)) \cup (4\delta D(\beta_2), \infty)$, *the boundary value problem (2.6) has a unique solution.*

Proof. Note that

$$D'(\beta) = \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt + \beta \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt.$$

The existence of β_1 follows from that $g' \geq v_0 > 0$ and $|g''|$ is bounded.

From $g'(s) = v(g(s))$, we have

$$\frac{dg}{v(g)} = ds \quad \text{ or } \quad \int_0^{g(z)} \frac{dr}{v(r)} = z.$$

Therefore,

$$z g''(z) = z v_r(g(z)) g'(z) = v_r(g(z)) v(g(z)) \int_0^{g(z)} \frac{1}{v(r)} dr.$$

If $v(r) = 0$ for large r in (1.5), then $\lim_{z \to +\infty} (g'(z) + z g''(z)) = v_0 > 0$. For the other case in (1.5),

$$\begin{aligned}
\lim_{z \to +\infty} (g'(z) + z g''(z)) =& v_0 + \lim_{z \to \infty} \left(v_r(g(z)) v(g(z)) \int_0^{g(z)} \frac{1}{v(r)} dr \right) \\
=& v_0 - \lim_{g \to \infty} \frac{v(g) v_r^2(g)/v_{rr}(g)}{v_r^2(g)/v_{rr}(g) + v(g)} = v_0 > 0.
\end{aligned}$$

One checks that, for any continuous function $q(t)$,

$$\lim_{\beta\to\infty}\int_0^1 \frac{q(\beta t)}{\sqrt{1-t}}dt = 2\lim_{t\to\infty} q(t)$$

if the latter limit exists. Therefore,

$$\lim_{\beta\to\infty} D'(\beta) = \lim_{\beta\to\infty}\int_0^1 \frac{g'(\beta t)+\beta t g''(\beta t)}{\sqrt{1-t}}dt = 2\lim_{z\to+\infty}(g'(z)+zg''(z)) > 0.$$

The existence of $\beta_2 > 0$ with the desired property follows directly. $\square$

Example. We end this section with an example of $v(r)$ for which $\bar{u} = 4\delta D(\beta)$ is a cubic-like function. We set $\delta = 1$ and choose a piecewise viscosity function

$$v(r) = \begin{cases} 1, & 0 \le r < 1 \\ (1+9(r-1)^8)^{-1}, & 1 \le r < 2 \\ 0.1, & 2 \le r. \end{cases} \tag{2.15}$$

This cubic $\bar{u}(\alpha)$ has a local maximum $\bar{u}_{\max} \approx 12.84$ and local minimum $\bar{u}_{\min} \approx 10.98$, Figure 2. For $\bar{u} \in (u_{\min}, u_{\max})$ there are three steady-state solutions which bifurcate from a unique solution as $\bar{u}$ is varied across the local extrema.

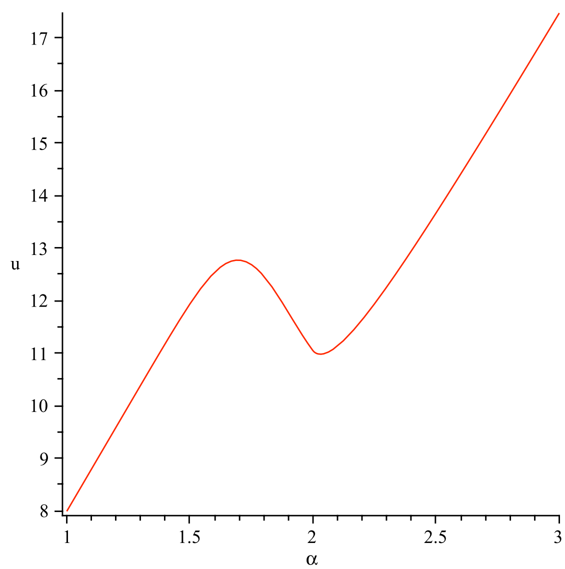

Figure 2: *A cubic-like $\bar{u}(\beta)$ for $v(r)$ in the example. Note that the horizontal axis is labeled by $\alpha = g(\beta)$ instead of β.*

3 Linear stability for small $\bar{u}$

In this section, we use energy methods to establish the linear stability of steady-states with small $\bar{u}$.

Let $(u^*, r^*) = (u^*(y), r^*(y))$ be a steady-state of the problem (1.3) and (1.4) with $u^*(1) = \bar{u}$. The linearization of the problem (1.3) and (1.4) along (u^*, r^*) is

$$U_t = \big(v(r^*)U_y + u^*_y v_r(r^*)R\big)_y, \quad R_t = \delta R_{yy} + U_y, \tag{3.16}$$

with $U(t,0) = U(t,1) = R(t,0) = R(t,1) = 0$.

Theorem 3.1. *For small $\bar{u}$, (u^*, r^*) is linearly exponentially stable in L^2; more precisely, for $K = \pi^2\delta^{-1}v_0^{-1}$, if $\bar{u}$ is small enough, then there exists $\rho > 0$ such that*

$$\int_0^1 (KU^2(t,y) + R^2(t,y))dy \le e^{-\rho t}\int_0^1 (KU^2(0,y) + R^2(0,y))dy.$$

Proof. By the Poincare inequality, we have, for R with $R(0) = R(1) = 0$,

$$\int_0^1 R^2(y)dy \le \frac{1}{\pi^2}\int_0^1 R_y^2(y)dy.$$

It follows from (2.7) that $M^2(\beta) \le \bar{u}v_1$ so that

$$u^*_y(y) = \frac{M^2(\beta)}{v(r^*(y))} \le \frac{v_1}{v_0}\bar{u}, \quad |u^*_y(y)v_r(r^*(y))| \le \frac{\bar{u}v_1\|v_r\|_{L^\infty}}{v_0}.$$

Multiply the U-equation by KU, R-equation by R, and integrate over $[0,1]$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (KU^2 + R^2)dy &= -\int_0^1 \big(Kv(r^*)U_y^2 + Ku^*_y v_r(r^*)RU_y + \delta R_y^2 - RU_y\big)\,dy \\ &\le -\int_0^1 \big(Kv_0U_y^2 + \delta R_y^2\big)\,dy + \frac{K\bar{u}v_1\|v_r\|_{L^\infty} + v_0}{v_0}\int_0^1 |RU_y|dy.\end{aligned}$$

By Young's inequality and the Poincare inequality,

$$\begin{aligned}\int_0^1 |RU_y|dy \le& \frac{\delta\pi^2}{2}\int_0^1 R^2dy + \frac{1}{2\pi^2\delta}\int_0^1 U_y^2dy \\ \le& \frac{\delta}{2}\int_0^1 R_y^2dy + \frac{1}{2\pi^2\delta}\int_0^1 U_y^2dy.\end{aligned}$$

It is clear that, for small $\bar{u}$,

$$Kv_0 > \frac{K\bar{u}v_1\|v_r\|_{L^\infty} + v_0}{2\pi^2\delta v_0} \quad \text{and} \quad \delta > \frac{K\bar{u}v_1\|v_r\|_{L^\infty} + v_0}{2v_0}\delta.$$

Thus, there exists $\rho > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (KU^2+R^2)dy \leq & -\frac{\rho}{2}\int_0^1 \frac{1}{\pi^2}(KU_y^2+R_y^2)dy \\ \leq & -\frac{\rho}{2}\int_0^1 (KU^2+R^2)dy.\end{aligned}$$

Hence, by Gronwall's inequality,

$$\int_0^1 (KU^2(t,y)+R^2(t,y))dy \leq e^{-\rho t}\int_0^1 (KU^2(0,y)+R^2(0,y))dy.$$

This establishes the L^2 linear stability of steady states with small $\bar{u}$. $\square$

4 Eigenvalues and bifurcations of steady-states

In view of the existence and multiplicity result (Theorem 2.2), steady-states of the boundary value problem (1.3) and (1.4) cannot be uniquely parameterized by $\bar{u}$ in general. We thus parameterize steady-states by the parameter β with $\bar{u}(\beta) = 4\delta D(\beta)$ and examine the spectral stability of steady-states as β varies.

It follows from the previous section that steady-states associated to small $\bar{u}$ are linearly stable. As we increase β, there are two possibilities for the steady-state to loss its stability: one is that a zero eigenvalue is created and the other is a pair of pure imaginary eigenvalues. In this section, we focus on stability changes of steady-states due to bifurcations of zero eigenvalues. The basic tool for this investigation is an Evans or a Wronskian type function.

4.1 Eigenvalue problem and an Evans function

For $\beta > 0$, let $(u,r) = (u(y;\beta), r(y;\beta))$ be the steady-state with $\bar{u} = 4\delta D(\beta)$ defined in (2.14). In view of the linearized system (3.16), the eigenvalue problem associated to this steady-state is the system

$$\left(v(r)U_y + u_y v_r(r)R\right)_y = \lambda U, \quad \delta R_{yy} + U_y = \lambda R \tag{4.17}$$

with the boundary condition

$$U(0) = R(0) = 0, \quad U(1) = R(1) = 0. \tag{4.18}$$

Alternatively, we can set

$$P = v(r)U_y + u_y v_r(r)R \ \text{ and } \ Q = \delta R_y + U,$$

and rewrite system (4.17) into a system of first order equations

$$U' = \frac{1}{v(r)}P - \frac{u_y v_r(r)}{v(r)}R, \quad P' = \lambda U, \quad R' = \frac{1}{\delta}Q - \frac{1}{\delta}U, \quad Q' = \lambda R, \tag{4.19}$$

where prime denotes the derivative with respect to y. Setting $Z = (U, P, R, Q)$, system (4.19) has the compact form

$$Z' = A(y; \lambda, \beta)Z, \tag{4.20}$$

where

$$A(y; \lambda, \beta) = \begin{pmatrix} 0 & \frac{1}{v(r(y;\beta))} & -\frac{u_y(y;\beta)v_r(r(y;\beta))}{v(r(y;\beta))} & 0 \\ \lambda & 0 & 0 & 0 \\ -\frac{1}{\delta} & 0 & 0 & \frac{1}{\delta} \\ 0 & 0 & \lambda & 0 \end{pmatrix}.$$

For any given $\beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$, let $Z_j(y; \lambda, \beta)$ for $j = 1, 2, 3, 4$ be the solutions of (4.19) with

$$Z_1(0; \lambda, \beta) = Z_3(1; \lambda, \beta) = e_2 = (0, 1, 0, 0),$$
$$Z_2(0; \lambda, \beta) = Z_4(1; \lambda, \beta) = e_4 = (0, 0, 0, 1)$$

so that Z_1 and Z_2 are linearly independent solutions and satisfy the boundary condition at $y = 0$, and Z_3 and Z_4 are linearly independent solutions and satisfy the boundary condition at $y = 1$.

Set

$$E(y; \lambda, \beta) = \det(Z_1(y; \lambda, \beta), Z_2(y; \lambda, \beta), Z_3(y; \lambda, \beta), Z_4(y; \lambda, \beta)). \tag{4.21}$$

We have

Lemma 4.1. *The function E is independent of y and is smooth in (λ, β).*

Proof. The claim follows from that

$$E(y; \lambda, \beta) = \exp\left\{\int_0^y \mathrm{tr}A(\tau; \lambda, \beta)d\tau\right\} E(0; \lambda, \beta)$$

and $\mathrm{tr}A(\tau; \lambda, \beta) = 0$. $\square$

We thus denote $E(y; \lambda, \beta)$ by $E(\lambda, \beta) : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ and refer to it as the *Evans function* of the eigenvalue problem (4.20). Evans function was widely used to study point spectrum of linearization along special solutions, such as various wave solutions, of systems of PDEs (see, for example, [8, 1, 11, 12, 5, 9, 6]) and the corresponding spectral problem is defined typically on the whole space. For the problem at hand, the eigenvalue problem is a boundary value problem but the idea for the construction of an Evans function is the same.

Lemma 4.2. *A number $\lambda \in \mathbb{C}$ is an eigenvalue if and only if $E(\lambda, \beta) = 0$.*

Proof. Suppose λ is an eigenvalue. Then there exists a nonzero solution $Z(y) = Z(y;\lambda,\beta) \neq 0$ of the boundary value problem (4.19). Let $Z(0) = (0, c_1, 0, c_2)$ and $Z(1) = (0, c_3, 0, c_4)$ for some c_i's, not all zeros. Since $Z(0) = c_1 Z_1(0;\lambda,\beta) + c_2 Z_2(0;\lambda,\beta)$ and $Z(1) = c_3 Z_3(1;\lambda,\beta) + c_4 Z_4(1;\lambda,\beta)$, one has

$$Z(y) = c_1 Z_1(y;\lambda,\beta) + c_2 Z_2(y;\lambda,\beta) = c_3 Z_3(y;\lambda,\beta) + c_4 Z_4(y;\lambda,\beta).$$

Therefore, $E(\lambda,\beta) = 0$. On the other hand, if $E(\lambda,\beta) = 0$, then

$$c_1 Z_1(y;\lambda,\beta) + c_2 Z_2(y;\lambda,\beta) + c_3 Z_3(y;\lambda,\beta) + c_4 Z_4(y;\lambda,\beta) = 0$$

for some c_i's, not all zeros. Since Z_1 and Z_2 are linearly independent, and Z_3 and Z_4 are linearly independent, it cannot happen that $c_1 = c_2 = 0$ or $c_3 = c_4 = 0$. Therefore,

$$Z(y) := c_1 Z_1(y;\lambda,\beta) + c_2 Z_2(y;\lambda,\beta) = -c_3 Z_3(y;\lambda,\beta) - c_4 Z_4(y;\lambda,\beta)$$

is a nonzero solution of the boundary value problem (4.19), and hence, the number λ is an eigenvalue. □

4.2 Zero eigenvalue and its bifurcation for $\lambda \in \mathbb{R}$.

In system (2.6) for the steady-states of (1.3), we introduce $p = v(r)u_y$ and $q = \delta r_y + u$. System (2.6) becomes

$$u_y = \frac{1}{v(r)}p, \quad p_y = 0, \quad r_y = \frac{1}{\delta}q - \frac{1}{\delta}u, \quad q_y = 0. \tag{4.22}$$

It can be checked directly that

Lemma 4.3. *System (4.22) has three integrals given by*

$$H_1 = p, \quad H_2 = q, \quad H_3 = \frac{1}{2}(q-u)^2 + \delta f(r)p.$$

When $\lambda = 0$, system (4.19) of eigenvalue problems is reduced to

$$U' = \frac{1}{v(r)}P - \frac{u_y v_r(r)}{v(r)}R, \quad P' = 0, \quad R' = \frac{1}{\delta}Q - \frac{1}{\delta}U, \quad Q' = 0, \tag{4.23}$$

which is nothing but the linearization of system (4.22) along the solution $z = (u, p, r, q)$ of (4.22). We have

Lemma 4.4. *System (4.23) has three integrals $G_j = \langle \nabla H_j(z), Z\rangle$:*

$$G_1 = P,\ G_2 = Q,\ G_3 = -(q-u)U + \delta f(r)P + \frac{\delta p}{v(r)}R + (q-u)Q.$$

Proof. One can verify the statement directly. In general, if $H(z)$ is an integral for a nonlinear system $z'(t) = F(z)$, then its linearization $Z' = DF(z(t))Z$ along a solution $z(t)$ has an integral given by $G = \langle \nabla H(z(t)), Z \rangle$. $\square$

As a consequence, we have

Lemma 4.5. *The principal fundamental matrix solution $\Phi(y)$ at $y = 0$ of system (4.23) is*

$$\Phi(y) = \begin{pmatrix} U_1(y) & U_2(y) & U_3(y) & U_4(y) \\ 0 & 1 & 0 & 0 \\ R_1(y) & R_2(y) & R_3(y) & R_4(y) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned}
U_1(y) =& \frac{v_1}{v(r(y))} + \frac{r'(0)}{v(r(y))} \int_0^y v_r(t)dt, \\
U_2(y) =& \frac{1}{v(r(y))} \int_0^y (v_r(r(t))f(r(t)) + 1)dt, \\
U_3(y) =& -\frac{u'(0)}{v(r(y))} \int_0^y v_r(r(t))dt, \\
U_4(y) =& -\frac{1}{\delta v(r(y))} \int_0^y u(t)v_r(r(t))dt = 1 - U_1(y), \\
R_1(y) =& \frac{r'(y)}{u'(y)} U_1(y) - \frac{r'(0)}{u'(y)} = \frac{r'(y)}{u'(0)} - \frac{r'(0)}{u'(y)} + \frac{r'(0)r'(y)}{v_1 u'(0)} \int_0^y v_r(r(t))dt, \\
R_2(y) =& \frac{r'(y)}{u'(y)} U_2(y) - \frac{f(r(y))}{u'(y)}, \\
R_3(y) =& \frac{r'(y)}{u'(y)} U_3(y) + \frac{u'(0)}{u'(y)}, \\
R_4(y) =& \frac{r'(y)}{u'(y)} U_4(y) + \frac{u(y)}{\delta u'(y)} = -R_1(y).
\end{aligned}$$

Proof. We construct only the second column of $\Phi(y)$ and the other columns can be found similarly. Suppose that $(U, P, R, Q)^T$ is a solution of (4.23) with with the initial condition e_2. It follows from Lemma 4.4 that, for all y,

$$P(y) = G_1(0) = 1, \; Q(y) = G_2(0) = 0, \; -\delta r'U + \delta f(r) + \delta u'R = G_3(0) = 0.$$

Substituting $R = \frac{r'}{u'}U - \frac{f(r)}{u'}$ into the U-equation of (4.23), we get

$$vU' + v_r r'U = v_r f(r) + 1.$$

Therefore,

$$U = \frac{1}{v} \int_0^y (v_r f(r) + 1) \, dt.$$

Hence,

$$R = \frac{r'}{u'}U - \frac{f(r)}{u'}.$$

This completes the proof. $\square$

Recall that $\bar{u}(\beta) = 4\delta D(\beta)$ in (2.14) where $D(\beta)$ is defined in (2.9).

Theorem 4.6. *The number $\lambda = 0$ is an eigenvalue associated to $\beta_* > 0$ if and only if $\bar{u}'(\beta_*) = 0$ (or equivalently, $D'(\beta_*) = 0$).*

Theorem 4.6 follows from Lemma 4.2 and

Proposition 4.7. *For $\beta > 0$, $E(0,\beta) = -8\beta^2 \bar{u}'(\beta)/\bar{u}^2(\beta)$.*

Proof. Recall the definition of $Z_j(y;\lambda,\beta)$, for $j = 1,2,3,4$, given next to system (4.20). Denote $Z_j^0(y) = Z_j(y;0,\beta)$ for simplicity.

It follows from Lemmas 4.5 and 4.1, (2.12), (2.13), (2.14) and $u'(1) = M^2/v_1$ that,

$$\begin{aligned} E(0,\beta) &= \det(Z_1^0(1), Z_2^0(1), Z_3^0(1), Z_4^0(1)) = \det(\Phi(1)e_2, \Phi(1)e_4, e_2, e_4) \\ &= \frac{r'(1) - r'(0)}{u'(1)} U_2(1) = -\frac{8\beta}{\bar{u}} \left(\int_0^1 v_r(r(t)) f(r(t)) dt + 1 \right). \end{aligned} \tag{4.24}$$

Using the symmetry of $r(y)$ with respect to $y = 1/2$ established in the proof of Theorem 2.2 and expression (2.12) and a number of substitutions, we have

$$\begin{aligned} \int_0^1 v_r(r(t)) f(r(t)) dt &= \frac{\sqrt{2\delta}}{M} \int_0^\alpha \frac{v_r(r) f(r)}{\sqrt{f(\alpha) - f(r)}} dr \\ &= \frac{\sqrt{2\delta}}{M} \int_0^\beta \frac{s v_r(g(s)) g'(s)}{\sqrt{\beta - s}} ds \\ &= \frac{\sqrt{2\delta}\beta^{\frac{3}{2}}}{M} \int_0^1 \frac{t v_r(g(\beta t)) g'(\beta t)}{\sqrt{1-t}} dt \\ &= \frac{\sqrt{2\delta}\beta^{\frac{3}{2}}}{M} \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt = \frac{4\delta\beta^2}{\bar{u}} \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt. \end{aligned} \tag{4.25}$$

In the second to last step, we have used the relation $g''(s) = v_r(g(s))g'(s)$ from $g'(s) = v(g(s))$ (see Lemma 2.1).

Recall that

$$\bar{u}'(\beta) = 4\delta \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt + 4\delta\beta \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt. \tag{4.26}$$

Substitute (4.25) into (4.24) and use (2.14) and (4.26) to get

$$\begin{aligned} E(0,\beta) &= -\frac{8\beta^2}{\bar{u}^2} \left(4\delta\beta \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt + \frac{\bar{u}(\beta)}{\beta} \right) \\ &= -\frac{8\beta^2}{\bar{u}^2} \left(4\delta\beta \int_0^1 \frac{t g''(\beta t)}{\sqrt{1-t}} dt + 4\delta \int_0^1 \frac{g'(\beta t)}{\sqrt{1-t}} dt \right) \\ &= -\frac{8\beta^2 \bar{u}'(\beta)}{\bar{u}^2(\beta)}. \end{aligned}$$

This completes the proof. $\square$

In general,

Lemma 4.8. *If, for some positive integer k, $\bar{u}'(\beta_*) = \cdots = \bar{u}^{(k)}(\beta_*) = 0$, then*

$$\frac{\partial^j E}{\partial \beta^j}(0, \beta_*) = 0 \;\; \text{for} \;\; j < k \;\; \text{and} \;\; \frac{\partial^k E}{\partial \beta^k}(0, \beta_*) = -\frac{8\beta_*^2}{\bar{u}^2(\beta_*)}\bar{u}^{(k+1)}(\beta_*).$$

The main technical result is

Proposition 4.9. *If β_* is a critical point of $\bar{u}(\beta)$, then*

$$E_\lambda(0, \beta_*) = \frac{16\delta\beta_*^3}{\bar{u}^3} L(\beta_*)$$

where

$$\begin{aligned} L(\beta) =& \delta \left(\int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}} d\tau \right)^{-1} \Delta - \int_0^1 g'(\beta\tau) \left(1 - \sqrt{1-\tau}\right) F(\tau, \beta) d\tau \\ & - \left(\int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}} d\tau \right)^{-1} \int_0^1 g'(\beta\tau)\sqrt{1-\tau} G(\tau, \beta) F(\tau, \beta) d\tau, \end{aligned}$$

where

$$F(\tau, \beta) = \int_0^\tau t g'(\beta t)(1-t)^{-3/2} dt, \quad G(\tau, \beta) = \int_0^\tau g'(\beta t)(1-t)^{-3/2} dt,$$

$$\Delta = \int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}} d\tau \int_0^1 \sqrt{1-\tau} F(\tau, \beta) d\tau - \int_0^1 \sqrt{1-\tau} G(\tau, \beta) F(\tau, \beta) d\tau.$$

It then follows that

Corollary 4.10. *Fix $v(r)$ and let β_* be a critical point of $\bar{u}(\beta)$. If $\Delta < 0$ or if $\Delta > 0$ but $\delta > 0$ is small enough, then $E_\lambda(0, \beta_*) < 0$.*

Remark 4.11. *We could not prove but suspect that $\Delta < 0$ is always true for decreasing functions $v(r)$ considered in the model problem.*

Before a proof of Proposition 4.9, we state our main result as a simple consequence.

If $E(0, \beta_*) = 0$ and $E_\lambda(0, \beta_*) \neq 0$, then, by the Implicit Function Theorem, there exists an $\eta > 0$ and a unique smooth function $\lambda(\beta)$ for $\beta \in (\beta_* - \eta, \beta_* + \eta)$ such that $\lambda(\beta_*) = 0$ and $E(\lambda(\beta), \beta) = 0$ for all $\beta \in (\beta_* - \eta, \beta_* + \eta)$. Then,

$$E_\beta(\lambda(\beta), \beta) + E_\lambda(\lambda(\beta), \beta)\lambda'(\beta) = 0$$

for all $\beta \in (\beta_* - \eta, \beta_* + \eta)$. In particular,

$$\lambda'(\beta_*) = -\frac{E_\beta(0, \beta_*)}{E_\lambda(0, \beta_*)} = \frac{8\beta_*^2 \bar{u}''(\beta_*)}{\bar{u}^2(\beta_*) E_\lambda(0, \beta_*)}. \tag{4.27}$$

As consequence of Corollary 4.10 and the above formula (4.27), we have

Theorem 4.12. *Assume the condition in Corollary 4.10 so that $E_\lambda(0,\beta_*) < 0$.*

(i) If β_* satisfies $\bar{u}'(\beta_*) = 0$ and $\bar{u}''(\beta_*) < 0$, then, for $\beta < \beta_*$ but close, there is exactly one negative eigenvalue close to zero (bifurcating from the zero eigenvalue of β_*); for $\beta > \beta_*$ but close, there is exactly one positive eigenvalue close to zero (bifurcating from the zero eigenvalue of β_*).

(ii) If β_* satisfies $\bar{u}'(\beta_*) = 0$ and $\bar{u}''(\beta_*) > 0$, then, for $\beta < \beta_*$ but close, there is exactly one positive eigenvalue close to zero (bifurcating from the zero eigenvalue of β_*); for $\beta > \beta_*$ but close, there is exactly one negative eigenvalue close to zero (bifurcating from the zero eigenvalue of β_*).

Remark 4.13. *In general, if $\bar{u}^{(k)}(\beta_*) = 0$ for $k = 1, \cdots, n$ and $\bar{u}^{(n+1)}(\beta_*) \neq 0$, then, from Lemma 4.8,*

$$\lambda^{(k)}(\beta_*) = 0 \ \ \textit{for} \ \ k = 1, 2, \cdots, n-1, \quad \lambda^{(n)}(\beta_*) = \frac{8\beta^2 \bar{u}^{(n+1)}(\beta_*)}{\bar{u}^2(\beta_*) E_\lambda(0,\beta_*)}.$$

One can then make conclusions on the bifurcation of the zero eigenvalue for $\beta \neq \beta_*$ but close to β_*.

5 Proof of Proposition 4.9

We start with some preparation.

Lemma 5.1. *$R_2(0) = R_2(1) = 0$ and $R_2(y) < 0$ for $y \in (0,1)$ and $R_2(y)$ is monotone for $y \in [0, 1/2)$.*

Proof. Note that $r_\beta(y;\beta_*) = p_\beta(\beta_*) R_2(y)$. Recall from (2.12) that, for $y \in (0, 1/2)$,

$$r'(y;\beta) = \sqrt{2\delta^{-1}} M(\beta) \left(\beta - f(r(y;\beta)\right)^{1/2},$$

and hence,

$$r'_\beta = a(y;\beta) r_\beta + \sqrt{\frac{2}{\delta(\beta - f(r))}} \left(\frac{M(\beta)}{2} + M_\beta(\beta)\left(\beta - f(r)\right) \right),$$

where

$$a(y;\beta) = -\frac{M(\beta)}{v(r)\sqrt{2\delta(\beta - f(r))}}.$$

Denote $\Psi(y)$ the principal fundamental matrix solution with system matrix $a(y;\beta)$. Then, noting that $r_\beta(0;\beta^*) = 0$,

$$r_\beta(y;\beta_*) = \int_0^y \Psi(y)\Psi^{-1}(t) \sqrt{\frac{2}{\delta(\beta_* - f(r))}} \left(\frac{M(\beta_*)}{2} + M_\beta(\beta_*)\left(\beta_* - f(r)\right) \right) dt.$$

It follows from

$$M(\beta)=\frac{\bar{u}(\beta)\beta^{-1/2}}{\sqrt{8\delta}}\quad\text{and}\quad M_\beta(\beta_*)=-\frac{\bar{u}(\beta_*)\beta_*^{-3/2}}{2\sqrt{8\delta}}$$

that, for $y\in(0,1/2)$,

$$\frac{M(\beta_*)}{2}+M_\beta(\beta_*)\left(\beta_*-f(r)\right)=\beta_*^{-3/2}f(r(y))>0.$$

Therefore, $r_\beta(y;\beta_*)>0$ for $y\in(0,1/2)$. The statement for $R_2(y)$ follows. $\square$

Lemma 5.2. *If β_* is a critical value of $\bar{u}(\beta)$, then*

$$\int_0^1\frac{g''(\beta_*t)}{\sqrt{1-t}}dt=\frac{\bar{u}(\beta_*)}{8\delta\beta_*^2}-\frac{v_1}{\beta_*}-\frac{\bar{u}(\beta_*)}{4\delta\beta_*^2}=-\frac{\bar{u}(\beta_*)}{8\delta\beta_*^2}-\frac{v_1}{\beta_*},$$

$$\int_0^1 v_r(r(t;\beta_*))dt=-\frac{1}{2\beta_*}-\frac{4\delta v_1}{\bar{u}(\beta_*)},\quad U_1(1)=-1-\frac{\bar{u}(\beta_*)}{4\delta\beta_*v_1},\quad R_1(1)=\frac{1}{\delta}.$$

Proof. It follows from the same line in (4.25) that, for any β,

$$\int_0^1 v_r(r(t;\beta))dt=\frac{4\delta\beta}{\bar{u}(\beta)}\int_0^1\frac{g''(\beta t)}{\sqrt{1-t}}dt.$$

If β_* is a critical value of $\bar{u}(\beta)$, then, from (2.14) and (4.26),

$$\frac{4\delta\beta_*^2}{\bar{u}(\beta_*)}\int_0^1\frac{tg''(\beta_*t)}{\sqrt{1-t}}dt=-1\quad\text{or}\quad\int_0^1\frac{tg''(\beta_*t)}{\sqrt{1-t}}dt=-\frac{\bar{u}(\beta_*)}{4\delta\beta_*^2}.$$

Now,

$$\begin{aligned}\int_0^1\frac{(1-t)g''(\beta_*t)}{\sqrt{1-t}}dt&=\int_0^1\sqrt{1-t}g''(\beta_*t)dt=\int_0^1\sqrt{1-t}\left(\frac{1}{\beta_*}g'(\beta_*t)\right)'dt\\&=-\frac{g'(0)}{\beta_*}+\frac{1}{2\beta_*}\int_0^t\frac{g'(\beta_*t)}{\sqrt{1-t}}dt=\frac{\bar{u}(\beta_*)}{8\delta\beta_*^2}-\frac{v_1}{\beta_*}.\end{aligned}$$

Thus,

$$\int_0^1\frac{g''(\beta_*t)}{\sqrt{1-t}}dt=\frac{\bar{u}(\beta_*)}{8\delta\beta_*^2}-\frac{v_1}{\beta_*}-\frac{\bar{u}(\beta_*)}{4\delta\beta_*^2}=-\frac{\bar{u}(\beta_*)}{8\delta\beta_*^2}-\frac{v_1}{\beta_*}.$$

Other statements follow immediately. $\square$

Lemma 5.3. *If β_* is a critical value of $\bar{u}(\beta)$, then $U_2(y)$ is odd and $R_2(y)$ is even with respect to $y=1/2$.*

Proof. We will show that $U_2(y)$ is odd with respect to $y = 1/2$ from which it follows by the relation in Lemma (4.5) that $R_2(y)$ is even. Fix $y \in [0, 1]$. Note that from the symmetry of $r(y)$ we have

$$\int_y^1 v_r(r(t))f(r(t))\,dt = \int_0^{1-y} v_r(r(t))f(r(t))\,dt.$$

Lemma (4.7) and the above give

$$\begin{aligned} 0 &= \int_0^1 \big(v_r(r(t))f(r(t)) + 1\big)\,dt \\ &= \int_0^y \big(v_r(r(t))f(r(t)) + 1\big)\,dt + \int_0^{1-y} \big(v_r(r(t))f(r(t)) + 1\big)\,dt. \end{aligned}$$

This implies $-U(y) = U(1 - y)$ proving the result. □

It follows from (4.21) and Lemma 4.1 that

$$\begin{aligned} E(\lambda, \beta) &= \det(Z_1(1; \lambda, \beta), Z_2(1; \lambda, \beta), Z_3(1; \lambda, \beta), Z_4(1; \lambda, \beta)) \\ &= \det(Z_1(1; \lambda, \beta), Z_2(1; \lambda, \beta), e_2, e_4). \end{aligned}$$

Hence,

$$\begin{aligned} E_\lambda(0, \beta_*) &= \det(Z_{1,\lambda}(1; 0, \beta_*), Z_2(1; 0, \beta_*), e_2, e_4) \\ &\quad + \det(Z_1(1; 0, \beta_*), Z_{2,\lambda}(1; 0, \beta_*), e_2, e_4). \end{aligned}$$

At $\lambda = 0$, $Z_1(1; 0, \beta_*) = e_2$ and hence,

$$E_\lambda(0, \beta_*) = \det(Z_{1,\lambda}(1; 0, \beta_*), Z_2(1; 0, \beta_*), e_2, e_4).$$

If we denote $Z_{1,\lambda}(1; 0, \beta_*) = (E_1, E_2, E_3, E_4)^T$, noting that

$$Z_2(1; 0, \beta_*) = (U_4(1), 0, R_4(1), 1)^T,$$

then

$$E_\lambda(0, \beta_*) = U_4(1)E_3 - R_4(1)E_1 = \frac{U_4(1)}{u'(1)}(u'(1)E_3 - r'(1)E_1) - \frac{\bar{u}}{\delta u'(1)}E_1. \tag{5.28}$$

It is known that $Z_{1,\lambda}(y) = Z_{1,\lambda}(y; 0, \beta_*)$ is a solution of

$$Z' = A(y; 0, \beta_*)Z + A_\lambda(y; 0, \beta_*)Z_1(y; 0, \beta_*) \tag{5.29}$$

with initial condition $Z(0) = 0$. Hence,

$$Z_{1,\lambda}(y) = \Phi(y) \int_0^y \Phi^{-1}(t)A_\lambda(t; 0, \beta_*)Z_1(t; 0, \beta_*)dt. \tag{5.30}$$

Using Lemma 4.5, one has

$$\Phi(1) = \begin{pmatrix} U_1(1) & 0 & U_3(1) & U_4(1) \\ 0 & 1 & 0 & 0 \\ R_1(1) & 0 & R_3(1) & R_4(1) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Phi^{-1}(y) = \begin{pmatrix} R_3 & U_3R_2 - U_2R_3 & -U_3 & U_3R_4 - U_4R_3 \\ 0 & 1 & 0 & 0 \\ -R_1 & U_2R_1 - U_1R_2 & U_1 & U_4R_1 - U_1R_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also,

$$A_\lambda(y; 0, \beta_*) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If we denote

$$\int_0^1 \Phi^{-1}(t) A_\lambda(t; 0, \beta_*) Z_1(t; 0, \beta_*) dt = (S_1, S_2, S_3, S_4)^T,$$

then

$$S_1 = \int_0^1 \left(U_2(U_3R_2 - U_2R_3) + R_2(U_3R_4 - U_4R_3)\right) dt, \ S_2 = \int_0^1 U_2 dt,$$
$$S_3 = \int_0^1 \left(U_2(U_2R_1 - U_1R_2) + R_2(U_4R_1 - U_1R_4)\right) dt, \ S_4 = \int_0^1 R_2 dt.$$

It then follows from (5.30) that

$$E_1 = U_1(1)S_1 + U_3(1)S_3 + U_4(1)S_4 \ \text{ and } \ E_3 = R_1(1)S_1 + R_3(1)S_3 + R_4(1)S_4.$$

Using the fact that $r'(0) = -r'(1) = \bar{u}/2\delta$, $u'(0) = u'(1)$, and the relations in Lemma (4.5) it is easy to show that

$$E_\lambda(0, \beta_*) = (r'(0)\delta)^{-1} \left(r'(0)S_1 - u'(0)S_3\right) - 2S_3.$$

For convenience we consider the integrands L_1 and L_3 of S_1 and S_3 respectivly. It follows from Lemma 4.5 that

$$U_4R_1 - U_1R_4 = R_1,$$

which gives

$$L_3 = U_2^2 R_1 - U_1U_2R_2 + R_1R_2$$

The expanded terms in L_1 are

$$\begin{aligned}U_3R_2 - U_2R_3 &= \frac{u'(0)}{r'(0)}\left(\frac{v_1}{v} - U_1\right)R_2 - U_2\left(\frac{r'}{r'(0)} - \frac{u'(0)}{r'(0)}R_1\right)\\ &= \frac{u'(0)v_1}{r'(0)v}R_2 - \frac{u'(0)}{r'(0)}U_1R_2 - \frac{r'}{r'(0)}U_2 + \frac{u'(0)}{r'(0)}R_1U_2\end{aligned}$$

and

$$\begin{aligned}U_3R_4 - U_4R_3 &= \frac{u'(0)}{r'(0)}\left(\frac{v_1}{v} - U_1\right)(-R_1) - (1-U_1)\left(\frac{r'}{r'(0)} - \frac{u'(0)}{r'(0)}R_1\right)\\ &= -\frac{u'(0)}{r'(0)}\frac{v_1}{v}R_1 + \frac{u'(0)}{r'(0)}R_1 + \frac{r'}{r'(0)}U_1 - \frac{r'}{r'(0)}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{r'(0)}{u'(0)}L_1 =&\frac{v_1}{v}U_2R_2 - U_1U_2R_2 - \frac{r'}{u'(0)}U_2^2 + U_2^2R_1 - \frac{v_1}{v}R_1R_2\\ &+ R_1R_2 + \frac{r'}{u'(0)}U_1R_2 - \frac{r'}{u'(0)}R_2\\ =&\frac{v_1}{v}U_2R_2 - \frac{r'}{u'(0)}U_2^2 - \frac{v_1}{v}R_1R_2 + \frac{r'}{u'(0)}U_1R_2 - \frac{r'}{u'(0)}R_2 + L_3.\end{aligned}$$

As a consequence of Lemma 5.3, after integration over the interval $[0,1]$, the first two terms $\frac{v_1}{v}U_2R_2$ and $u'(0)r'U_2^2$ will vanish. Thus, we drop these terms. It follows from Lemma 4.4 that $-r'(0) = -r'U_1 + u'R_1$, which gives the reduction

$$r'(0)L_1 = (r'(0) - r')\,R_2 + u'(0)L_3$$

Again we drop the term $r'R_2$, as it will vanish after integration, to obtain

$$r'(0)L_1 - u'(0)L_3 = r'(0)R_2 \tag{5.31}$$

Turning our attention back to L_3, since

$$\begin{aligned}\int_0^y r'(t)U_1(t)\,dt =& \int_0^y r'(0)\frac{d}{dt}\big(f(r(t))\big)\int_0^t v_r(r(s))\,ds + v_1\frac{d}{dt}[f(r(t))]\,dt\\ =&f(r(y))\left(r'(0)\int_0^y v_r(r(t))\,dt + v_1\right)\\ &- r'(0)\left(\int_0^y f(r(t))v_r(r(t)) + 1\,dt\right) + r'(0)y\\ =&v(r(y))f(r(y))U_1(y) - v(r(y))r'(0)U_2(y) + r'(0)y,\end{aligned}$$

after expanding $U_2R_1 - U_1R_2$ we have

$$\int_0^1 U_2^2R_1 - U_1U_2R_2\,dt = \int_0^1 U_2\int_0^t \frac{1}{v(r(s))}R_1(s)\,ds\,dt. \tag{5.32}$$

Finally, noting that $-\delta R_2(y) = \int_0^y U_2(t)\,dt$, we integrate the above expression by parts and combine with (5.31) to obtain

$$E_\lambda(0,\beta_*) = \frac{1}{\delta}\int_0^1 R_2(t)\,dt - 2\int_0^1 R_1(t)R_2(t)\,dt - 2\int_0^1 \frac{\delta}{v(r(t))}R_1(t)R_2(t)\,dt.$$

It is easy to check that, for any function $\phi(v)$ and $\psi(v) = v\phi(v)$,

$$\begin{aligned}\int_0^1 \phi(v(r))R_2 dy =& 2\int_0^{1/2}\phi(v(r))R_2 dy\\ =& \frac{2}{M^2}\int_0^{1/2}\phi(v(r))r'\int_0^y (v_r f+1)dtdy - \frac{2}{M^2}\int_0^{1/2} f(r)\psi(v(r))dy.\end{aligned}$$

We have

$$\begin{aligned}&\int_0^{1/2}\phi(v(r(y)))r'(y)\int_0^y (v_r f+1)dtdy\\ &= \int_0^\alpha \phi(v(p))\int_0^{r^{-1}(p)}(v_r f+1)dtdp\\ &= \frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^\alpha \phi(v(p))\int_0^p \frac{v_r(z)f(z)+1}{\sqrt{f(\alpha)-f(z)}}dzdp\\ &= \frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^\alpha \phi(v(p))\int_0^{f(p)} \frac{sg''(s)+g'(s)}{\sqrt{\beta-s}}dsdp\\ &= \frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^{\beta_*} \psi(g'(w))\int_0^{w} \frac{sg''(s)+g'(s)}{\sqrt{\beta_*-s}}dsdw\\ &= \frac{\sqrt{\delta}\beta_*^{3/2}}{\sqrt{2}M}\int_0^1 \psi(g'(\beta_*\tau))\int_0^\tau \frac{\beta_* tg''(\beta_* t)+g'(\beta_* t)}{\sqrt{1-t}}dtd\tau\\ &= \frac{\sqrt{\delta}\beta_*^{3/2}}{\sqrt{2}M}\int_0^1 \psi(g'(\beta_*\tau))\int_0^\tau \frac{(tg'(\beta_* t))_t}{\sqrt{1-t}}dtd\tau,\end{aligned}$$

and

$$\begin{aligned}\int_0^{1/2} f(r)\psi(v(r))dy =& \frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^\alpha \frac{f(p)\psi(v(p))}{\sqrt{f(\alpha)-f(z)}}dp\\ =& \frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^{\beta_*} \frac{sg'(s)\psi(g'(s))}{\sqrt{\beta_*-s}}ds\\ =& \frac{\sqrt{\delta}\beta_*^{3/2}}{\sqrt{2}M}\int_0^1 \frac{\tau g'(\beta_*\tau)\psi(g'(\beta_*\tau))}{\sqrt{1-\tau}}d\tau.\end{aligned}$$

Also,

$$\begin{aligned}\int_0^\tau \frac{(tg'(\beta_* t))_t}{\sqrt{1-t}}dt =& tg'(\beta_* t)(1-t)^{-1/2}|_0^\tau - \frac{1}{2}\int_0^\tau tg'(\beta_* t)(1-t)^{-3/2}dt\\ =& \frac{\tau g'(\beta_*\tau)}{\sqrt{1-\tau}} - \frac{1}{2}\int_0^\tau tg'(\beta_* t)(1-t)^{-3/2}dt.\end{aligned}$$

Therefore,

$$\int_0^1 \phi(v(r(y)))R_2(y)dy = -\frac{\sqrt{\delta}\beta_*^{3/2}}{\sqrt{2}M^3}\int_0^1 \psi(g'(\beta_*\tau))\int_0^\tau tg'(\beta_* t)(1-t)^{-3/2}dt\,d\tau.$$

Note that

$$\begin{aligned}
&\int_0^1 \frac{v+\delta}{v}R_1(y)R_2(y)dy\\
&=\int_0^1\left(\frac{r'}{u'(0)}-\frac{r'(0)}{u'}+\frac{r'(0)r'}{M^2}\int_0^y v_r dt\right)\frac{v+\delta}{v}R_2 dy\\
&=-\int_0^1 \frac{r'(0)}{u'}\frac{v+\delta}{v}R_2 dy+\frac{r'(0)}{M^2}\int_0^1 r'\int_0^y v_r dt\frac{v+\delta}{v}R_2 dy\\
&=-\frac{2r'(0)}{M^2}\int_0^{1/2}(v+\delta)R_2 dy+\frac{r'(0)}{M^2}\int_0^{1/2} r'\int_0^y v_r dt\frac{v+\delta}{v}R_2 dy\\
&\quad+\frac{r'(0)}{M^2}\int_{1/2}^1 r'\int_0^y v_r dt\frac{v+\delta}{v}R_2 dy\\
&=-\frac{r'(0)}{M^2}\int_0^1(v+\delta)R_2 dy-\frac{2r'(0)}{M^2}\int_0^{1/2} r'\int_y^{1/2} v_r dt\frac{v+\delta}{v}R_2 dy,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{1/2} r'\int_y^{1/2} v_r dt\frac{v+\delta}{v}R_2(y)dy =&\frac{1}{M^2}\int_0^{1/2}\frac{v+\delta}{v}r'r'\int_0^y(v_r f+1)dt\int_y^{1/2} v_r dt dy\\
&-\frac{1}{M^2}\int_0^{1/2}(v+\delta)fr'\int_y^{1/2} v_r dt dy\\
=:&\frac{1}{M^2}(I_1-I_2).
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 =&\frac{\sqrt{2}M}{\sqrt{\delta}}\int_0^\alpha \frac{v(p)+\delta}{v(p)}\sqrt{f(\alpha)-f(p)}\int_0^{r^{-1}(p)}(v_r f+1)dt\int_{r^{-1}(p)}^{1/2} v_r dt dp\\
=&\frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^\alpha \frac{v(p)+\delta}{v(p)}\sqrt{\beta_*-f(p)}\int_0^p \frac{v_r(z)f(z)+1}{\sqrt{\beta_*-f(z)}}dz\int_p^\alpha \frac{v_r(z)}{\sqrt{\beta_*-f(z)}}dz dp\\
=&\frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^\alpha \frac{v(p)+\delta}{v(p)}\sqrt{\beta_*-f(p)}\int_0^{f(p)} \frac{sg''(s)+g'(s)}{\sqrt{\beta_*-s}}ds\int_{f(p)}^{\beta_*} \frac{g''(s)}{\sqrt{\beta_*-s}}ds dp\\
=&\frac{\sqrt{\delta}}{\sqrt{2}M}\int_0^{\beta_*}(g'(q)+\delta)\sqrt{\beta_*-q}\int_0^q \frac{sg''(s)+g'(s)}{\sqrt{\beta_*-s}}ds\int_q^{\beta_*} \frac{g''(s)}{\sqrt{\beta_*-s}}ds\,dq,
\end{aligned}$$

and

$$
\begin{aligned}
I_2 =& \int_0^{\alpha} (v(p)+\delta) f(p) \int_{r^{-1}(p)}^{1/2} v_r dt\, dp \\
=& \frac{\sqrt{\delta}}{\sqrt{2}M} \int_0^{\alpha} (v(p)+\delta) f(p) \int_p^{\alpha} \frac{v_r(z)}{\sqrt{\beta_* - f(z)}} dz\, dp \\
=& \frac{\sqrt{\delta}}{\sqrt{2}M} \int_0^{\alpha} (v(p)+\delta) f(p) \int_{f(p)}^{\beta_*} \frac{g''(s)}{\sqrt{\beta_* - s}} ds\, dp \\
=& \frac{\sqrt{\delta}}{\sqrt{2}M} \int_0^{\beta_*} (g'(q)+\delta) g'(q) q \int_q^{\beta_*} \frac{g''(s)}{\sqrt{\beta_* - s}} ds\, dq.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_1 - I_2 =& -\frac{\sqrt{\delta}}{2\sqrt{2}M} \int_0^{\beta_*} (g'(q)+\delta)\sqrt{\beta_* - q} \int_0^q \frac{s g'(s)}{(\beta_* - s)^{3/2}} ds \int_q^{\beta_*} \frac{g''(s)}{\sqrt{\beta_* - s}} ds\, dq \\
=& -\frac{\sqrt{\delta}\beta_*^{5/2}}{2\sqrt{2}M} \int_0^1 (g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_0^{\tau} \frac{t g'(\beta_* t)}{(1-t)^{3/2}} dt \int_{\tau}^1 \frac{g''(\beta_* t)}{\sqrt{1-t}} dt\, d\tau.
\end{aligned}
$$

Set, as introduced in the statement of Proposition 4.9,

$$
F(\tau, \beta_*) = \int_0^{\tau} \frac{t g'(\beta_* t)}{(1-t)^{3/2}} dt.
$$

Then,

$$
\begin{aligned}
-\frac{\sqrt{2}M^3 \bar{u}}{8\sqrt{\delta}\beta_*^{5/2}} E_\lambda(0,\beta_*) =& \int_0^1 \left(\frac{\bar{u}}{8\beta_*\delta} + g'(\beta_*\tau) + \delta \right) g'(\beta_*\tau) F(\tau,\beta_*) d\tau \\
&+ \beta_* \int_0^1 (g'(\beta_*\tau)+\delta)\sqrt{1-\tau} F(\tau,\beta_*) \int_{\tau}^1 \frac{g''(\beta_* t)}{\sqrt{1-t}} dt\, d\tau.
\end{aligned}
$$

It follows from Lemma 5.2 that

$$
\begin{aligned}
\beta_*(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_{\tau}^1 \frac{g''(\beta_* t)}{\sqrt{1-t}} dt =& -(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \left(\frac{\bar{u}}{8\beta_*\delta} + v_1 \right) \\
&- \beta_*(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_0^{\tau} \frac{g''(\beta_* t)}{\sqrt{1-t}} dt.
\end{aligned}
$$

and

$$
\begin{aligned}
-\beta_*(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_0^{\tau} \frac{g''(\beta_* t)}{\sqrt{1-t}} dt =& -(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_0^{\tau} \frac{(g'(\beta_* t))_t}{\sqrt{1-t}} dt \\
=& -(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \left(\frac{g'(\beta_*\tau)}{\sqrt{1-\tau}} - v_1 - \frac{1}{2}\int_0^{\tau} g'(\beta_* t)(1-t)^{-3/2} dt \right) \\
=& -(g'(\beta_*\tau)+\delta) g'(\beta_*\tau) + v_1 (g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \\
&+ \frac{1}{2}(g'(\beta_*\tau)+\delta)\sqrt{1-\tau} \int_0^{\tau} g'(\beta_* t)(1-t)^{-3/2} dt.
\end{aligned}
$$

If we set

$$
\begin{aligned}
G(\tau,\beta_*) &= \int_0^\tau g'(\beta_* t)(1-t)^{-3/2}dt,\\
L(\beta_*) &= \frac{8\beta_*\delta}{\bar{u}}\frac{\sqrt{2}M^3\bar{u}}{8\sqrt{\delta}\beta_*^{5/2}}E_\lambda(0,\beta_*) = \frac{\sqrt{2\delta}M^3}{\beta_*^{3/2}}E_\lambda(0,\beta_*),
\end{aligned}
$$

Then,

$$
\begin{aligned}
L(\beta) = &\int_0^1 (g'(\beta\tau)+\delta)\sqrt{1-\tau}F(\tau,\beta)d\tau - \int_0^1 g'(\beta\tau)F(\tau,\beta)d\tau\\
&- \frac{4\beta\delta}{\bar{u}}\int_0^1 (g'(\beta\tau)+\delta)\sqrt{1-\tau}G(\tau,\beta)F(\tau,\beta)d\tau\\
=&\delta\left(\int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}}d\tau\right)^{-1}\Delta - \int_0^1 g'(\beta\tau)\left(1-\sqrt{1-\tau}\right)F(\tau,\beta)d\tau\\
&- \left(\int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}}d\tau\right)^{-1}\int_0^1 g'(\beta\tau)\sqrt{1-\tau}G(\tau,\beta)F(\tau,\beta)d\tau,
\end{aligned}
$$

where

$$
\Delta = \int_0^1 \frac{g'(\beta\tau)}{\sqrt{1-\tau}}d\tau\int_0^1 \sqrt{1-\tau}F(\tau,\beta)d\tau - \int_0^1 \sqrt{1-\tau}G(\tau,\beta)F(\tau,\beta)d\tau.
$$

This then completes the proof of Proposition 4.9.

6 Hysteresis: a numerical simulation of dynamic boundary conditions

Our bifurcation analysis of the zero eigenvalue shows the stability change of the steady-state when β crosses critical points of $\bar{u}(\beta)$. For a certain potential functions $v(r)$ (see the example at the end of Sect. 2), the function $\bar{u} = 4\delta D(\beta)$ is cubic-like and the condition in Corollary 4.10 holds. Assume we are in this case. Let $\bar{u}_1$ be the local maximum value and let $\bar{u}_2$ be the local minimum value. The stability result suggests the following scenario for a hysteresis: if we consider the dynamic boundary condition by letting $\bar{u}(t)$ increase in t slowly from small value to large value, then, for $t < t_1$ so that $\bar{u}(t_1) = \bar{u}_1$, the solution $(u(y,t), r(y,t))$ of (1.3) and (1.4) with $\bar{u} = \bar{u}(t)$ will behave closely to the left-branch of steady-states associated to $\bar{u} = \bar{u}(t)$ and, for $t > t_1$, the solution $(u(y,t), r(y,t))$ will behave closely to the steady-state associated to $\bar{u} = \bar{u}(t) > \bar{u}_1$ on the right-branch; if we now reverse the dynamic boundary condition by letting $\bar{u}(t)$ decreases slowly from large value to small value, then, for $t < t_2$ where t_2 is the first time so that $\bar{u}(t_2) = \bar{u}_2$, the solution $(u(y,t), r(y,t))$ will behave closely to the right-branch of steady-states associated to $\bar{u} = \bar{u}(t)$ and, for $t > t_2$, the solution $(u(y,t), r(y,t))$ will behave closely to the steady-state associated to $\bar{u} = \bar{u}(t) < \bar{u}_2$ on the left-branch. In particular, the two processes are not reversible to each other over the range $(\bar{u}_2, \bar{u}_1)$ of $\bar{u}$; that is, this problem possesses a hysteresis phenomenon. Although we could not justify this hysteresis rigorously, a numerical simulation provides a strong support.

For the numerical simulation, we consider two 'opposite' dynamic boundary conditions for (1.3) and (1.4) with

$$\bar{u} = \bar{u}_+(t) = \begin{cases} L, & t \in [0, T_1] \\ h(t), & t \in [T_1, T_2] \\ R, & t \in [T_2, T] \end{cases}$$

and its 'reverse'

$$\bar{u} = \bar{u}_-(t) = \begin{cases} R, & t \in [0, T_1] \\ h(T_1 + T_2 - t), & t \in [T_1, T_2] \\ L, & t \in [T_2, \infty) \end{cases}$$

where $L < \bar{u}_2 < \bar{u}_1 < R$, $T_2 \gg T_1 \gg 1$, and $h(t)$ is increasing with $h(T_1) = L$ and $h(T_2) = R$. So the first dynamic boundary condition $\bar{u} = \bar{u}_+(t)$ is slowly increasing in t and the other $\bar{u} = \bar{u}_-(t)$ slowly decreasing. For the first boundary condition $\bar{u} = \bar{u}_+(t)$, we choose the steady-state associated to boundary condition $\bar{u} = L$ as the initial condition and for the second the steady-state associated to boundary condition $\bar{u} = R$ as the initial condition. Snaps shots of the numerical simulation (u-component only) are provided in Figure 3 with the left set for $\bar{u} = \bar{u}_+(t)$ and the right for $\bar{u} = \bar{u}_-(t)$. It shows clearly that the two sets of figures are not 'reverse' to each other.

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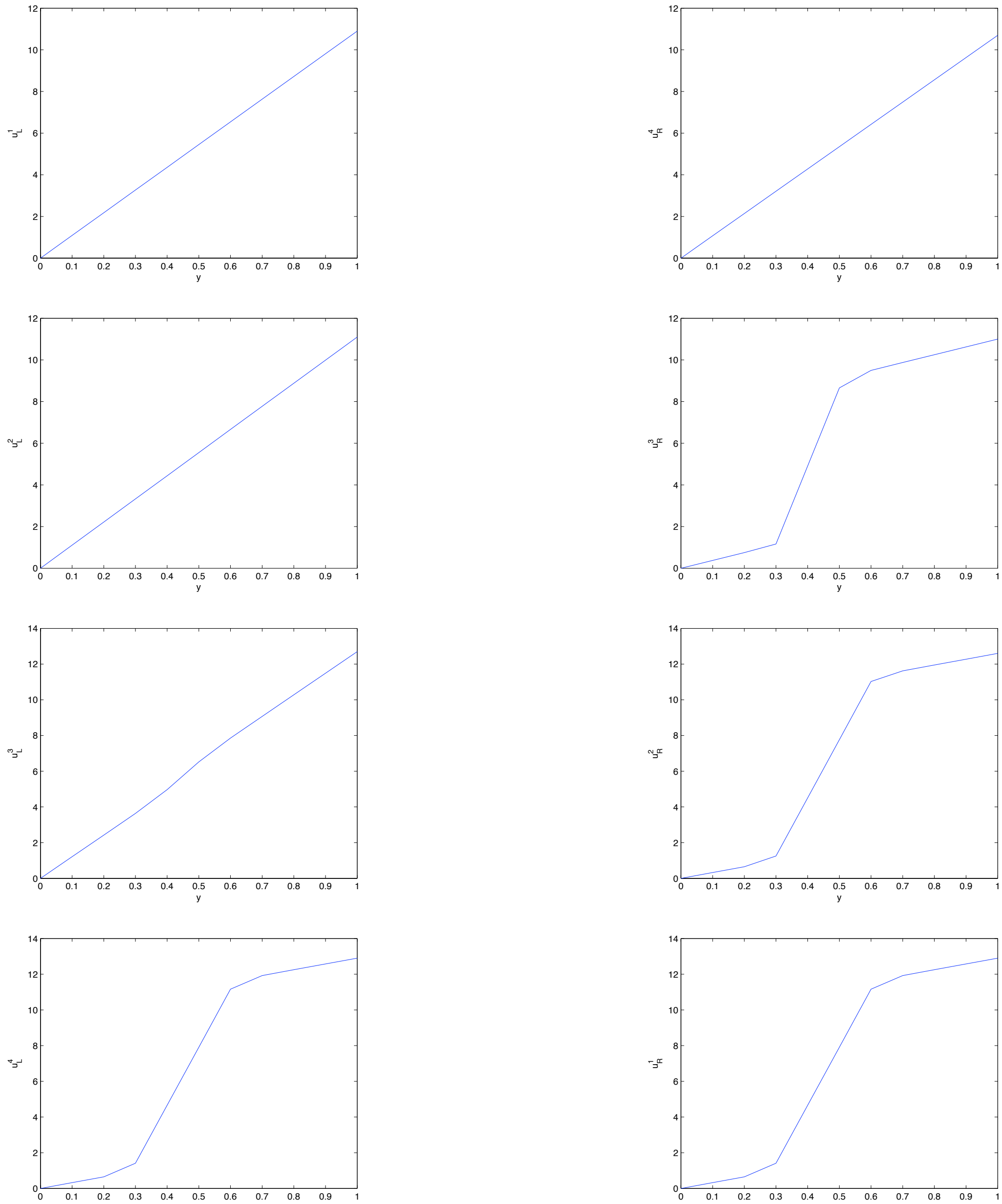

Figure 3: On the left hand side, beginning at the top, the right boundary condition $\bar{u}$ is slowly increased. When a value of $\bar{u}$ is near a critical point $\bar{u}_1 \approx 10.98$ or $\bar{u}_2 \approx 12.78$, we pause the boundary condition in order to converge to a steady state. The left hand side pauses at the values $u_L^1(1) = 10.6, u_L^2(1) = 11.1, u_L^3(1) = 12.7$, and $u_L^4(1) = 12.9$ and the right hand, beginning from the bottom, pauses at the values $u_R^1(1) = 12.9, u_R^2(1) = 12.6, u_R^3(1) = 11,$ and $u_R^4(1) = 10.7$.