Instability of low density supersonic waves of a viscous isentropic gas flow through a nozzle

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Abstract

In this work, we examine the stability of stationary non-transonic waves for viscous isentropic compressible flows through a nozzle with varying cross-section areas. The main result in this paper is, for small viscous strength, stationary supersonic waves with sufficiently low density are spectrally unstable; more precisely, we will establish the existence of positive eigenvalues for the linearization along such waves. The result is achieved via a center manifold reduction of the eigenvalue problem. The reduced eigenvalue problem is then studied in the framework of the Sturm-Liouville Theory.

1 Introduction

In this work, we examine the stability of some stationary solutions for the viscous compressible gas flows through a nozzle. The model system is

$$\begin{aligned}(a\rho)_t + (a\rho u)_x =& \varepsilon(a\rho_x)_x \\ (a\rho u)_t + (a\rho u^2)_x + a(P(\rho))_x =& \varepsilon(a(\rho u)_x)_x,\end{aligned} \tag{1}$$

where ρ, u, P and $a = a(x)$ are the density, velocity, pressure of the gas and the area of the cross section at x of the rotationally symmetric tube of the

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nozzle. The pressure P is assumed to be a given function of the density ρ. Our assumptions on the pressure P is:

$$\begin{aligned}
&\textit{For } \ \rho > 0,\ P(\rho) > 0 \textit{ and } \ P'(\rho) > 0;\\
&\lim_{\rho\to 0^+} \frac{P'(\rho)}{\rho} = \infty,\ \lim_{\rho\to 0^+} \rho^2 P'(\rho) = \lim_{\rho\to 0^+} \frac{\rho^8 P'''(\rho)}{P'(\rho)} = \lim_{\rho\to 0^+} \frac{\rho^9 P^{(4)}(\rho)}{P'(\rho)} = 0; \qquad (2)\\
&\textit{There exists } M > 0 \textit{ such that for } \ \rho > 0,\ \left|\frac{\rho P''(\rho)}{P'(\rho)}\right| \le M.
\end{aligned}$$

For polytropic gas, $P(\rho) = A\rho^\gamma$ with some constant $A > 0$ and $1 \le \gamma \le 5/3$ clearly satisfies the above assumption. Our assumption on $a(x)$ is

$$\lim_{x\to\pm\infty} a(x) = a_\pm > 0 \ \text{ and } \ \lim_{x\to\pm\infty} a_x(x) = \lim_{x\to\pm\infty} a_{xx}(x) = 0. \qquad (3)$$

The inviscid ($\varepsilon = 0$) system (1) is a well-known one-dimensional Euler equation describing the motion of isentropic compressible fluid through a narrow nozzle with variable cross-section area (see [1, 2, 3, 10, 11, 12, 13, 14, 16, 17, 18] etc.). The specific form of the viscous terms is not completely physical and should be regarded as artificial one. The study of the problem with the more physical viscosity is an ongoing project.

In [13], T. P. Liu studied global solutions of the initial value problem for general quasilinear strictly hyperbolic systems including the inviscid isentropic compressible gas flow:

$$w_t + f(w)_x = g(x, w).$$

Roughly speaking, it was shown that, for an initial data $w_0(x)$, if all eigenvalues $\lambda_j(w)$ of f_w are nonzero and the L^1-norm of g and g_w are small for w uniformly close to w_0, then a global solution exists and tends pointwise to a steady-state solution. For polytropic gas flow, the main assumptions become the flow at $t = 0$ is not close to transonic and that the total variation of the cross-section area $a(x)$ is sufficiently small.

T. P. Liu then focused on *transonic* waves of gas flow in a nozzle of varying area via the inviscid $\varepsilon = 0$ model (1) in [14]. Various types of solutions were shown to exist that demonstrated significant qualitative differences between a contracting nozzle (for example, $a_x(x) < 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0, 1)$) and an expanding nozzle (for example, $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin (0, 1)$). Asymptotic states along a nozzle that contracts and then expands ($a_x(x) < 0$ for $-1 < x < 0$, $a_x(x) > 0$ for $0 < x < 1$ and $a_x(x) \equiv 0$ for $x \notin [-1, 1]$) are also examined to exhibit a number of interesting phenomena including the choking phenomenon.

In [10], S.-B. Hsu and T. P. Liu studied a singular Sturm-Liouville problem

$$\varepsilon u_{xx} = f(x,u)_x - h(x)g(u) \tag{4}$$

and applied the result to the viscous steady-state problem of (1) with more physical viscosity terms. Assuming the nozzle is uniform outside a bounded portion and is either contracting or expanding otherwise, they reformulated the problem as a boundary value problem and gave a detailed analysis on the existence, multiplicity and uniqueness of solutions. Viewing solutions of the boundary value problem as steady-states of the corresponding reaction-diffusion equation

$$u_t = \varepsilon u_{xx} - f(x,u)_x + h(x)g(u),$$

stability results were also obtained.

Recently, in [6, 7], M. Hong, et al. studied the steady-state problem of system (1) with different choices of the viscosity and for expanding-contracting ($a_x > 0$ for $x < 0$ and $a_x < 0$ for $x > 0$) and contracting-expanding ($a_x < 0$ for $x < 0$ and $a_x > 0$ for $x > 0$) nozzles. They applied the geometric singular perturbation theory to provide a rather complete description of stationary waves for both the inviscid and viscous systems, and found classes of new types of transonic waves. In particular, transonic waves from subsonic to supersonic are constructed for contracting-expanding nozzle. In [8], the maximal sub-to-super transonic wave is shown to be linearly stable.

In this paper, we will conduct a case study and examine the stability of stationary non-transonic waves – simplest steady states ([6, 7]). Our main result is that supersonic waves with sufficiently low density are spectrally *unstable* as long as $a_x(x)$ changes sign; more precisely, we will establish the existence of positive eigenvalues for the linearization along such waves (see Theorem 5.1). Our result is directly relevant to the stability result in [13] where T.P. Liu constructed global solutions for quasilinear hyperbolic systems and studied their asymptotic behaviors. In particular, under some conditions, he established the stability of supersonic and subsonic waves. There seems to be a contradiction between T.P. Liu's stability result with our instability results. Our explanation lies in the following two reasons: first of all, the conditions under which the stability result of T.P. Liu do not hold for supersonic waves with sufficiently low density, secondly, T.P. Liu considered hyperbolic systems and we have the viscosity terms. An interesting observation is that, a certain form of viscosity might cause stable waves for inviscid flows to be unstable.

Our instability result relies on a center manifold reduction of the eigenvalue problem. The reduced eigenvalue problem turns out to be a quadratic eigenvalue problem and it is then studied via the Sturm-Liouville theory. The method of center manifold reduction in stability study has been applied by others (see, e.g., [15]).

The rest of the paper is organized as follows. We recall, in Section 2, the relevant existence results on stationary waves from [6, 7]. In Section 3, we set up the eigenvalue problem and make a center manifold reduction of the eigenvalue problem. Section 4 focuses on the $\varepsilon = 0$ limiting reduced eigenvalue problem and provides a symmetric structure of eigenvalues for symmetric nozzles. We then show that supersonic waves with sufficiently low density are spectrally unstable in Section 5 to complete the paper.

2 Steady-state problem

We recall the relevant existence result on stationary non-transonic waves from [6, 7] with a slight extension. To distinguish the variables from those of the linearization, we use $\bar{\rho}$, etc. for the stationary solutions. We introduce new variables

$$\bar{w} = \varepsilon a(\bar{\rho}\bar{u})_x - a(\bar{\rho}\bar{u}^2 + P(\bar{\rho})), \quad \bar{v} = \varepsilon a\bar{\rho}_x - a\bar{\rho}\bar{u}. \tag{5}$$

The steady-state system of (1) becomes,

$$\begin{cases} \varepsilon\bar{\rho}_x = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{v}_x = 0, \\ \varepsilon\bar{u}_x = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_x = -a_x P(\bar{\rho}). \end{cases} \tag{6}$$

Note that $\bar{v} = \bar{v}(\varepsilon)$ is constant. We also introduce a variable $\eta \in (-1, 1)$ via $\eta_x = 1 - \eta^2$. It is obvious that $\eta(x)$ is increasing in x and $\eta(\pm\infty) = \pm 1$.

This well-known trick allows one to replace the x-variable in $a(x)$ with $x(\eta)$ so that system (6) becomes an autonomous system

$$\begin{cases} \varepsilon\bar{\rho}_x = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \varepsilon\bar{u}_x = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_x = -a_x P(\bar{\rho}), \\ \eta_x = 1 - \eta^2. \end{cases} \tag{7}$$

System (7) is the so-called *slow system*. In terms of the fast time $\xi = x/\varepsilon$, the corresponding *fast system* is

$$\begin{cases} \bar{\rho}_\xi = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{u}_\xi = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_\xi = -\varepsilon a_x P(\bar{\rho}), \\ \eta_\xi = \varepsilon(1 - \eta^2). \end{cases} \tag{8}$$

The limiting slow and fast systems are, respectively,

$$\begin{cases} 0 = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ 0 = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_x = -a_x P(\bar{\rho}), \\ \eta_x = 1 - \eta^2, \end{cases} \tag{9}$$

and

$$\begin{cases} \bar{\rho}_\xi = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{u}_\xi = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_\xi = 0, \\ \eta_\xi = 0. \end{cases} \tag{10}$$

The slow manifold $\mathcal{Z}_0$ is given by

$$\mathcal{Z}_0 = \big\{ a\bar{\rho}\bar{u} + \bar{v} = 0 \ \text{ and } \ \bar{w} - \bar{v}\bar{u} + aP(\bar{\rho}) = 0 \big\}.$$

For the linearization of system (10) along $\mathcal{Z}_0$, the eigenvalues r_1 and r_2 in the directions transversal to $\mathcal{Z}_0$ are those of

$$\begin{pmatrix} \bar{u} & \bar{\rho} \\ \bar{\rho}^{-1}P'(\bar{\rho}) & \bar{u} \end{pmatrix},$$

that is,

$$r_1(\bar{\rho}, \bar{u}, \bar{w}, \eta) = \bar{u} - \sqrt{P'(\bar{\rho})} \ \text{ and } \ r_2(\bar{\rho}, \bar{u}, \bar{w}, \eta) = \bar{u} + \sqrt{P'(\bar{\rho})}.$$

Recall that $\sqrt{P'(\bar\rho)}$ is the sound speed. Thus, $(\bar\rho, \bar u)$ is called a supersonic (resp. sonic or subsonic) state at x if $\bar u(x) > \sqrt{P'(\bar\rho(x))}$ (resp. $\bar u(x) = \sqrt{P'(\bar\rho(x))}$ or $\bar u(x) < \sqrt{P'(\bar\rho(x))}$). Set

$$\begin{aligned}
\mathcal{Z}_0^u =&\{(\bar\rho, \bar u, \bar w, \eta) \in \mathcal{Z}_0 :\ \bar u > \sqrt{P'(\bar\rho)},\ \bar\rho > 0,\ \eta \in [-1, 1]\ \},\\
\mathcal{Z}_0^s =&\{(\bar\rho, \bar u, \bar w, \eta) \in \mathcal{Z}_0 :\ \bar u < \sqrt{P'(\bar\rho)},\ \bar\rho > 0,\ \eta \in [-1, 1]\ \},\\
T =&\{(\bar\rho, \bar u, \bar w, \eta) \in \mathcal{Z}_0 :\ \bar u = \sqrt{P'(\bar\rho)},\ \bar\rho > 0,\ \eta \in [-1, 1]\ \}.
\end{aligned}$$

Then $\mathcal{Z}_0 = \mathcal{Z}_0^s \cup T \cup \mathcal{Z}_0^u$. The portion $\mathcal{Z}_0^s$ is (normally) saddle and consists of subsonic states, $\mathcal{Z}_0^u$ is (normally) repelling and consists of supersonic states, and T is the set of turning points and consists of sonic states.

For the dynamics of the limiting slow flow on $\mathcal{Z}_0$, we differentiate

$$\bar u = -\bar v a^{-1} \bar\rho^{-1} \ \text{ and } \ \bar w = -\bar v^2 a^{-1} \bar\rho^{-1} - aP(\bar\rho)$$

with respect to x and use system (9) to get

$$-aP'(\bar\rho)\bar\rho_x + \bar v^2 a^{-2} a_x \bar\rho^{-1} + \bar v^2 a^{-1} \bar\rho^{-2} \bar\rho_x = 0.$$

Thus, the limiting slow dynamics on $\mathcal{Z}_0$ can be represented by the system

$$\bar\rho_x = \frac{\bar v^2 a^{-3} a_x \bar\rho^{-1}}{P'(\bar\rho) - \bar v^2 a^{-2} \bar\rho^{-2}}, \quad \eta_x = 1 - \eta^2. \tag{11}$$

It can be checked directly (see [6, 7]) that system (11) has an integral

$$I(\bar\rho, \eta) = E(\bar\rho) + \frac{\bar v^2}{2a^2(x(\eta))\bar\rho^2} \ \text{ where } \ E(\bar\rho) = \int_{\bar\rho_0}^{\bar\rho} \frac{P'(s)}{s}\, ds. \tag{12}$$

By an inviscid non-transonic wave, we mean, for each fixed $\bar v < 0$, a solution $(\bar\rho(x), \bar u(x), \bar w(x), \eta(x))$ of system (9) so that $r_j \neq 0$ for $j = 1, 2$. By a viscous profile of $(\bar\rho(x), \bar u(x), \bar w(x), \eta(x))$, we mean, for $\bar v(\varepsilon) \to \bar v$ as $\varepsilon \to 0$, a solution $(\bar\rho(x;\varepsilon), \bar u(x;\varepsilon), \bar w(x;\varepsilon), \eta(x))$ of system (7) so that

$$(\bar\rho(x;\varepsilon), \bar u(x;\varepsilon), \bar w(x;\varepsilon), \eta(x)) \to (\bar\rho(x), \bar u(x), \bar w(x), \eta(x))$$

as $\varepsilon \to 0$ in L^1_{loc}.

The following result can be readily obtained. For more complete results and proofs, we refer the readers to the papers [6, 7].

Theorem 2.1. *Fix $\bar{v} < 0$. For any $\bar{\rho}_-$ and $\bar{\rho}_+$, there is an inviscid non-transonic wave $(\bar{\rho}(x), \bar{u}(x))$ for system (9) or (11) with $a(x)\bar{\rho}(x)\bar{u}(x) + \bar{v} = 0$ and $\bar{\rho}(x) \to \bar{\rho}_\pm$ as $x \to \pm\infty$ if and only if*

$$\int_{\bar{\rho}_0}^{\bar{\rho}_-} \frac{P'(s)}{s}\, ds + \frac{\bar{v}^2}{2a_-^2\bar{\rho}_-^2} = \int_{\bar{\rho}_0}^{\bar{\rho}_+} \frac{P'(s)}{s}\, ds + \frac{\bar{v}^2}{2a_+^2\bar{\rho}_+^2}$$

and the level set L lies entirely in either $\mathcal{Z}_0^s$ or $\mathcal{Z}_0^u$ where

$$L = \{(\bar{\rho}, \eta) : I(\bar{\rho}, \eta) = I(\bar{\rho}_-, -1) = I(\bar{\rho}_+, 1)\}.$$

Any such a non-transonic wave admits viscous profiles.

In the rest, we consider the stability of viscous profiles for system (1).

3 The eigenvalue problem and a center manifold reduction

With the same new variables introduced in (5), we rewrite system (1) as

$$\begin{cases} \varepsilon\rho_x = a^{-1}v + \rho u, \\ v_x = (a\rho)_t, \\ \varepsilon u_x = a^{-1}\rho^{-1}(w - vu + aP(\rho)), \\ w_x = (a\rho u)_t - a_x P(\rho). \end{cases} \tag{13}$$

Let $(\bar{\rho}(x;\varepsilon), \bar{v}(x;\varepsilon), \bar{u}(x;\varepsilon), \bar{w}(x;\varepsilon))$ be a stationary wave for $\varepsilon \geq 0$. In the following, we will drop the argument $(x;\varepsilon)$. It should be clear from the context when $\varepsilon = 0$ and when $\varepsilon > 0$.

The eigenvalue problem for the stationary wave is

$$\begin{cases} \varepsilon\rho_x = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_x = \lambda a\rho, \\ \varepsilon u_x = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad\quad +a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_x = \lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho. \end{cases} \tag{14}$$

We will treat this problem as a singularly perturbed problem. In terms of the fast scale $\xi = x/\varepsilon$, it becomes

$$\begin{cases} \rho_\xi = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_\xi = \varepsilon\lambda a\rho, \\ u_\xi = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad +a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_\xi = \varepsilon\left(\lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho\right). \end{cases} \tag{15}$$

Next, we augment the eigenvalue problem (15) with the steady-state system (7) to obtain an autonomous problem as the following:

$$\begin{cases} \rho_\xi = a^{-1}v + \bar{\rho}u + \bar{u}\rho, \\ v_\xi = \varepsilon\lambda a\rho, \\ u_\xi = -a^{-1}\bar{\rho}^{-2}\left(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})\right)\rho \\ \qquad +a^{-1}\bar{\rho}^{-1}\left(w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho\right), \\ w_\xi = \varepsilon\left(\lambda a\bar{\rho}u + \lambda a\bar{u}\rho - a_x P'(\bar{\rho})\rho\right), \\ \bar{\rho}_\xi = \bar{v}a^{-1} + \bar{\rho}\bar{u}, \\ \bar{v}_\xi = 0, \\ \bar{u}_\xi = a^{-1}\bar{\rho}^{-1}(\bar{w} - \bar{v}\bar{u} + aP(\bar{\rho})), \\ \bar{w}_\xi = -\varepsilon a_x P(\bar{\rho}), \\ \eta_\xi = \varepsilon(1-\eta^2). \end{cases} \tag{16}$$

The phase space of this system is $\mathbb{R}^9$ with the variable $(\rho, v, u, w, \bar{\rho}, \bar{v}, \bar{u}, \bar{w}, \eta)$.

Viewing system (16) as a singularly perturbed autonomous system, the slow manifold $\mathcal{S}_0$ is given by

$$\begin{aligned} \mathcal{S}_0 =& \Big\{ \bar{\rho}\bar{u} + \bar{v}a^{-1} = 0, \quad \bar{w} - \bar{v}\bar{u} + aP(\bar{\rho}) = 0, \\ & \quad a^{-1}v + \bar{\rho}u + \bar{u}\rho = 0, \quad w - \bar{v}u - \bar{u}v + aP'(\bar{\rho})\rho = 0 \Big\} \\ =& \Big\{ \bar{u} = -\bar{v}a^{-1}\bar{\rho}^{-1}, \quad \bar{w} = -\bar{v}^2 a^{-1}\bar{\rho}^{-1} - aP(\bar{\rho}), \\ & \quad u = -a^{-1}\bar{\rho}^{-1}v + \bar{v}a^{-1}\bar{\rho}^{-2}\rho, \quad w = \frac{\bar{v}^2}{a\bar{\rho}^2}\rho - \frac{2\bar{v}}{a\bar{\rho}}v - aP'(\bar{\rho})\rho \Big\}. \end{aligned} \tag{17}$$

Note that $\mathcal{S}_0$ contains equilibria of the limiting fast system (16) with $\varepsilon = 0$ and $\dim \mathcal{S}_0 = 5$.

The linearized matrix of system (16) with $\varepsilon = 0$ at each point on $\mathcal{S}_0$ has the form

$$\begin{pmatrix} R & * & * \\ 0 & R & * \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$R = \begin{pmatrix} \bar{u} & a^{-1} & \bar{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\rho}^{-1}P'(\bar{\rho}) & -a^{-1}\bar{u}\bar{\rho}^{-1} & -a^{-1}\bar{v}\bar{\rho}^{-1} & a^{-1}\bar{\rho}^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

The sets of eigenvalues are $\{0, r_1, r_2\}$ where the zero eigenvalue with multiplicity five corresponds to the dimension of the slow manifold $\mathcal{S}_0$ and where

$$r_1 = \bar{u} - \sqrt{P'(\bar{\rho})}, \quad r_2 = \bar{u} + \sqrt{P'(\bar{\rho})} \tag{19}$$

each with multiplicity two.

We will consider only non-transonic stationary wave $(\bar{\rho}, \bar{v}, \bar{u}, \bar{w})$, that is, for any x, $\bar{u}(x) \neq \sqrt{P'(\bar{\rho}(x))}$. In this case, both r_1 and r_2 are non-zero, and hence, $\mathcal{S}_0$ is normally hyperbolic. The normally hyperbolic invariant manifold theory then implies that, for $\varepsilon > 0$ small, $\mathcal{S}_0$ persists and the perturbed slow manifold has the form

$$\begin{aligned} \mathcal{S}_\varepsilon = \Big\{ \bar{u} &= -\bar{v}a^{-1}\bar{\rho}^{-1} + \varepsilon G, \quad \bar{w} = -\frac{\bar{v}^2}{a\bar{\rho}} - aP(\bar{\rho}) + \varepsilon F, \\ u &= -a^{-1}\bar{\rho}^{-1}v + \bar{v}a^{-1}\bar{\rho}^{-2}\rho + \varepsilon(h_1 v + h_2 \rho), \\ w &= \frac{\bar{v}^2}{a\bar{\rho}^2}\rho - \frac{2\bar{v}}{a\bar{\rho}}v - aP'(\bar{\rho})\rho + \varepsilon(F_1 v + F_2 \rho) \Big\}, \end{aligned} \tag{20}$$

where the argument for the functions F, G, F_j's and h_j's is $(\bar{\rho}, \bar{v}, \eta, \varepsilon)$.

Remark 3.1. The normally hyperbolic invariant manifold theory ([4, 9]) requires the invariant manifold to be bounded or compact. The slow manifold $\mathcal{S}_0$ is not. We can certainly restrict the $(\bar{\rho}, \bar{v}, \bar{u}, \bar{w}, \eta)$-component to a bounded neighborhood of the steady-states but the eigenvalue problem needs the whole linear (ρ, v, u, w)-component. To get around this non-boundeness, one can view the first four linear equations in system (16) as defined in the projective space of $\mathbb{R}^4$ so that homogeneous (ρ, v, u, w)-component lies in the compact projective space. The normally hyperbolic invariant manifold theory can be applied and, afterwards, one can return back to the original setting. This also explains why the extra terms in u and w for $\mathcal{S}_\varepsilon$ take the special form. $\square$

The idea is then to reduce the eigenvalue problem (16) onto $\mathcal{S}_\varepsilon$. To do so, we first substitute (20) into (7) and (14), and after some tedious algebra, we find, up to $O(\varepsilon)$, that

$$\begin{aligned} G =& \frac{\bar{v}^2 a^{-2} a_x \bar{\rho}^{-1}}{a\bar{\rho}\left(P'(\bar{\rho}) - \bar{v}^2 a^{-2}\bar{\rho}^{-2}\right)}, \quad h_1 = \frac{\lambda + 2\bar{v}a^{-2}a_x\bar{\rho}^{-1} + 2\bar{v}a^{-1}\bar{\rho}^{-1}G}{a\bar{\rho}\left(P'(\bar{\rho}) - \bar{v}^2 a^{-2}\bar{\rho}^{-2}\right)}, \\ h_2 =& -\frac{2\lambda\bar{v}\bar{\rho}^{-1} + \bar{v}^2 a^{-2} a_x \bar{\rho}^{-2} + \bar{v}^2 a^{-1}\bar{\rho}^{-2} G + a\bar{\rho}P''(\bar{\rho})G + aP'(\bar{\rho})G}{a\bar{\rho}\left(P'(\bar{\rho}) - \bar{v}^2 a^{-2}\bar{\rho}^{-2}\right)}. \end{aligned} \tag{21}$$

On the center manifold $\mathcal{S}_\varepsilon$, up to $O(\varepsilon^2)$, the first four equations in system (16) are reduced to a system of two equations,

$$\begin{cases} \rho_\xi = va^{-1} + \bar{\rho}(-va^{-1}\bar{\rho}^{-1} + \bar{v}a^{-1}\bar{\rho}^{-2}\rho + \varepsilon(h_1 v + h_2\rho)) \\ \qquad +(-\bar{v}a^{-1}\bar{\rho}^{-1} + \varepsilon G)\rho \\ \quad = \varepsilon\ \bar{\rho}h_1 v + \varepsilon(\bar{\rho}h_2 + G)\rho, \\ v_\xi = \varepsilon\lambda a\rho. \end{cases} \tag{22}$$

If we return to the x-variable, the latter system becomes

$$\rho_x = f(\bar{\rho}, \bar{v}; \varepsilon)v + g(\bar{\rho}, \bar{v}; \varepsilon)\rho, \quad v_x = \lambda a\rho \tag{23}$$

where $f = \bar{\rho}h_1 + O(\varepsilon)$ and $g = G + \bar{\rho}h_2 + O(\varepsilon)$. System (23) is referred to as *the reduced eigenvalue problem* via the center manifold reduction.

4 The limiting eigenvalue problem with $\varepsilon = 0$

In this section, we will consider the limiting eigenvalue problem of (23) with $\varepsilon = 0$:

$$\rho_x = f(\bar{\rho}(x), \bar{v}; 0)v + g(\bar{\rho}(x), \bar{v}; 0)\rho, \quad v_x = \lambda a\rho \tag{24}$$

where $\bar{\rho} = \bar{\rho}(x)$ is the ρ-component of the inviscid stationary wave.

System (24) can be cast as

$$v_{xx} - \left(\frac{a_x}{a} + g\right)v_x = \lambda afv. \tag{25}$$

Setting $y = v\mathrm{e}^{-\frac{1}{2}\int(\frac{a_x}{a}+g)}$, we have

$$y_{xx} - \left(\frac{1}{4}\left(\frac{a_x}{a}+g\right)^2 - \frac{1}{2}\left(\frac{a_x}{a}+g\right)_x\right) y = \lambda a f y. \tag{26}$$

If we separate f and g into terms without λ and with λ as $f = f_1 + \lambda f_2$ and $g = g_1 + \lambda g_2$, then equation (26) becomes

$$y_{xx} - \left(\frac{1}{4}q^2 - \frac{1}{2}q_x\right) y = \lambda w_1 y + \lambda^2 w_2 y, \tag{27}$$

where

$$q = \frac{a_x}{a} + g_1,\ w_1 = af_1 + \frac{1}{2}\left(\frac{a_x}{a}+g_1\right) g_2 - \frac{1}{2}g_{2x},\ w_2 = af_2 + \frac{1}{4}g_2^2.$$

It follows from (21) that

$$q = \frac{a_x}{a}\left(1 - \frac{\bar{v}^2 a^{-2}\bar{\rho}^5(P' + \bar{\rho}P'') + \bar{v}^4 a^{-4}\bar{\rho}^3}{(\bar{\rho}^2 P' - \bar{v}^2 a^{-2})^3}\right), \tag{28}$$

$$w_1 = -\frac{\bar{v}^2 a^{-4} a_x (a\bar{u}\bar{\rho} + a^{-2})\bar{\rho}^{-1}P'(\bar{\rho})}{\bar{\rho}^2 P'(\bar{\rho}) - \bar{v}^2 a^{-2}}, \tag{29}$$

and

$$w_2 = \frac{\bar{\rho}^4 P'(\bar{\rho})}{(\bar{\rho}^2 P'(\bar{\rho}) - \bar{v}^2 a^{-2})^2} > 0. \tag{30}$$

For later use, we collect some estimates

Lemma 4.1. *As $\bar{\rho} \to 0^+$, $\bar{\rho}^{-1}P'(\bar{\rho}) \to \infty$, and*

$$\begin{aligned} &q^2 + |q_x| + |q_{xx}| + w_2 + |w_{2x}| = o(\bar{\rho}^{-1}P'(\bar{\rho})), \\ &w_1 = a_x \bar{\rho}^{-1}P'(\bar{\rho})O(1), \quad w_{1x} = O(\bar{\rho}^{-1}P'(\bar{\rho})). \end{aligned}$$

Proof. It follows from the displays (28), (29) and (30), and (11) that

$$
\begin{aligned}
q =& \frac{a_x}{a} + O(P' + \bar{\rho}P'')\bar{\rho}^5 + O(\bar{\rho}^3),\\
q_x =& \left(\frac{a_x}{a}\right)_x + O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''')\bar{\rho}^5\\
& + O(P' + \bar{\rho}P'')\bar{\rho}^8 P'' + O(\bar{\rho}^3),\\
q_{xx} =& \left(\frac{a_x}{a}\right)_{xx} + O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''' + \bar{\rho}^3 P^{(4)})\bar{\rho}^5\\
& + O(P' + \bar{\rho}P'' + \bar{\rho}^2 P''')\bar{\rho}^8 P'' + O(\bar{\rho}^3),\\
w_1 =& a_x \bar{\rho}^{-1} P' O(1), \quad w_{1x} = O(\bar{\rho}^{-1}P' + P''),\\
w_2 =& O(\bar{\rho}^4 P'), \quad w_{2x} = O(P' + \bar{\rho}P'')\bar{\rho}^4.
\end{aligned}
$$

The conclusion is then a direct consequence of the assumption (2). □

We end this section with two simple results.

Lemma 4.2. *If $a_x \le 0$ and $a_x \ne 0$, then, for every subsonic wave $(\bar{\rho}, \bar{u})$, all eigenvalues have negative real parts. If $a_x \ge 0$ and $a_x \ne 0$, then, for every supersonic wave $(\bar{\rho}, \bar{u})$, all eigenvalues have negative real parts.*

Proof. Let λ be an eigenvalue and let $y(x)$ be an eigenfunction associated to λ. We multiply the conjugate $\bar{y}$ of y on (27) and integrate over $(-\infty, \infty)$ to get

$$
-\int |y_x|^2 - \frac{1}{4}\int q^2 |y|^2 + \frac{1}{2}\int q_x y\bar{y} = \lambda \int w_1 |y|^2 + \lambda^2 \int w_2 |y|^2.
$$

An application of integration by parts for the third term on the left gives

$$
-\int \left(y_x + \frac{1}{2}qy\right)\left(\bar{y}_x + \frac{1}{2}q\bar{y}\right) = \lambda \int w_1 |y|^2 + \lambda^2 \int w_2 |y|^2.
$$

Note that $w_2 > 0$ from (30). In view of (29), we have $w_1 \ge 0$ and $w_1 \ne 0$ if either $a_x \le 0$, $a_x \ne 0$ and $(\bar{\rho}, \bar{u})$ is subsonic or $a_x \ge 0$, $a_x \ne 0$ and $(\bar{\rho}, \bar{u})$ is supersonic. The conclusion then follows directly. □

Lemma 4.3. *Assume $a(x)$ is symmetric with respect to $x = 0$ and $(\bar{\rho}, \bar{u})$ is a symmetric non-transonic steady-state. Then, eigenvalues occur in pairs $\lambda, -\lambda$.*

Proof. It is easy to check that $\frac{1}{4}q^2 - \frac{1}{2}q_x$ is even, w_2 is even, but w_1 is odd. Suppose $y(x)$ is an eigenfunction associated to λ. Let $\hat{y}(x) = y(-x)$. Then $\hat{y}_{xx}(x) = y_{xx}(-x)$ and

$$\begin{aligned}\hat{y}_{xx}(x) - \left(\frac{q^2(x)}{4} - \frac{q_x(x)}{2}\right)\hat{y} =& y_{xx}(-x) - \left(\frac{q^2(-x)}{4} - \frac{q_x(-x)}{2}\right)y(-x)\\ =&\lambda w_1(-x)y(-x) + \lambda^2 w_2(-x)y(-x)\\ =&(-\lambda)w_1(x)\hat{y}(x) + (-\lambda)^2 w_2(x)\hat{y}(x).\end{aligned}$$

Thus, $-\lambda$ is also an eigenvalue. □

5 Instability of low density steady-states

We will consider now the case where $a_x(x)$ changes sign or simply the set $\{x : a_x(x) < 0\}$ is not empty and show that supersonic waves with sufficiently low density are unstable. Our main result is

Theorem 5.1. *Assume the set $\{x : a_x(x) < 0\}$ is not empty. Then there is a constant $\kappa > 0$ so that, if $(\bar{\rho}, \bar{u})$ is a stationary supersonic wave and $|\bar{\rho}| \le \kappa$, then $(\bar{\rho}, \bar{u})$ is spectrally unstable.*

Theorem 5.1 is a direct consequence of Lemma 5.3 to be proved later. We first recall a basic result from the Sturm-Liouville Theory.

Proposition 5.2. *(Theorem 5.2 in [5]) Let $r(t) > 0$ be continuous and of bounded variation on $[x_0, x_1]$. If $y(x) \neq 0$ is a real-valued solution of*

$$y_{xx} + r(x)y = 0$$

and N is the number of its zeros on $(x_0, x_1]$, then

$$\left|N - \frac{1}{\pi}\int_{x_0}^{x_1} \sqrt{r(x)}dx\right| \le 1 + \frac{1}{4\pi}\int_{x_0}^{x_1} \frac{|r_x(x)|}{r(x)}dx.$$

Now, for any real λ, let

$$r(x;\lambda) = -\frac{1}{4}q^2(x) + \frac{1}{2}q_x(x) - \lambda w_1(x) - \lambda^2 w_2(x).$$

Then equation (27) becomes

$$y_{xx} + r(x;\lambda)y = 0. \tag{31}$$

It can be expressed as

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ -r(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{32}$$

In view of the assumption in (3) and the displays (28), (29) and (30), we have

$$\lim_{x\to\pm\infty} q(x) = \lim_{x\to\pm\infty} w_1(x) = 0,$$

and hence,

$$r_\pm(\lambda) =: \lim_{x\to\pm\infty} r(x;\lambda) = -\lambda^2 \lim_{x\to\pm\infty} w_2(x) = \frac{-\lambda^2\bar{\rho}_\pm^4 P'(\bar{\rho}_\pm)}{\left(\bar{\rho}_\pm^2 P'(\bar{\rho}_\pm) - \bar{v}^2 a_\pm^{-2}\right)^2}.$$

Thus, if λ is real and $\lambda \neq 0$, then $r_\pm(\lambda) < 0$, and hence,

$$\begin{pmatrix} 0 & 1 \\ -r_\pm(\lambda) & 0 \end{pmatrix}$$

has two real non-zero eigenvalues $\pm\sqrt{|r_\pm(\lambda)|}$ with associated eigenvectors

$$v_\pm^s(\lambda) = \frac{1}{1+|r(\lambda)|}\left(1, -\sqrt{|r_\pm(\lambda)|}\right)^T, \; v_\pm^u(\lambda) = \frac{1}{1+|r(\lambda)|}\left(1, \sqrt{|r_\pm(\lambda)|}\right)^T.$$

The unit vector $v_-^s(\lambda)$ (resp. $v_-^u(\lambda)$) is the stable (resp. unstable) eigenvector of system (32) at $x = -\infty$. The unit vector $v_+^s(\lambda)$ (resp. $v_+^u(\lambda)$) is the stable (resp. unstable) eigenvector of system (32) at $x = \infty$.

Therefore, for any real $\lambda \neq 0$, there exists a unique solution $(y_\lambda(x), z_\lambda(x))^T$ of (32) such that

$$|(y_\lambda(0), z_\lambda(0))^T| = 1 \quad \text{and} \quad \frac{(y(x;\lambda), z(x;\lambda))^T}{|(y(x;\lambda), z(x;\lambda))^T|} \to v_-^u(\lambda) \quad \text{as} \quad x \to -\infty.$$

In particular, $y_\lambda(x) \neq 0$ is a solution of (31) and $y_\lambda(x) \to 0$ as $x \to -\infty$.

Let $N_\lambda = N_\lambda(y_\lambda(x))$ be the number of zeros of $y_\lambda(x)$ on $(-\infty, \infty)$. It follows from the asymptotic hyperbolicity of (32) that N_λ is finite. Note that N_λ is essentially twice the number of full clockwise rotations of the solution $(y_\lambda(x), z_\lambda(x))^T$ of system (32) for $x \in (-\infty, \infty)$.

Lemma 5.3. *Assume the set $\{x : a_x(x) < 0\}$ is nonempty. For any $\Lambda > 0$ and any positive integer n, there exists $\kappa > 0$ such that, if $\bar{\rho} \le \kappa$, then there are at least n eigenvalues in $(0, \Lambda)$.*

Proof. From the assumption, there is $\delta > 0$ so that $U(\delta) = \{x : a_x(x) < -\delta\}$ is open and non-empty. For fixed $\Lambda > 0$, let

$$I(\delta) = \{x \in U(\delta) : r(x; \Lambda) > 0\}.$$

Due to (29) and Lemma 4.1, for $x \in U(\delta)$, $\lim_{\bar{\rho}(x) \to 0} r(x; \Lambda) = \infty$ and there exists $K > 0$ such that $\dfrac{|r_x(x; \Lambda)|}{r(x; \Lambda)} \le K$. Hence, for any positive integer n, there exists $\kappa > 0$ such that, if $\bar{\rho} \le \kappa$, then, $I(\delta) = U(\delta)$ and

$$\frac{1}{\pi} \int_{I(\delta)} \sqrt{r(x; \Lambda)} dx - 1 - \frac{1}{4\pi} \int_{I(\delta)} \frac{|r_x(x; \Lambda)|}{r(x; \Lambda)} dx > n.$$

Proposition 5.2 implies that

$$N_\Lambda|_{I(\delta)} \ge \frac{1}{\pi} \int_{I(\delta)} \sqrt{r(x; \Lambda)} dx - 1 - \frac{1}{4\pi} \int_{I(\delta)} \frac{|r_x(x; \Lambda)|}{r(x; \Lambda)} dx > n.$$

Therefore, for $\bar{\rho} \le \kappa$, we have $N_\Lambda \ge N_\Lambda|_{I(\delta)} > n$.

Now, fix any $\bar{\rho} \le \kappa$. If $\lambda = 0$, one verifies that

$$y(x) = \exp\left\{ -\frac{1}{2} \int_0^x q(s) ds \right\}$$

is a solutions of (27) so that $N(y(x)) = 0$. By continuous dependence of solutions on parameter λ and the asymptotic hyperbolicity of system (32), it follows that, for any integer $1 \le j \le n$, there exists $\lambda_j \in (0, \Lambda)$ and a solution $y_j(x) \ne 0$ of equation (31) with $\lambda = \lambda_j$ such that $y_j(x) \to 0$ as $x \to \pm\infty$ and $N(y_j(x)) = j$. Each λ_j is then an eigenvalue with an eigenfunction $y_j(x)$. Since, for $i \ne j$, $N(y_i(x)) \ne N(y_j(x))$, we have $y_i(x)$ and $y_j(x)$ are linearly independent, and hence, $\lambda_i \ne \lambda_j$. □

References

[1] C.M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity mehtod. *Arch. Rational Mech. Anal.* **52** (1973), 1-9.

[2] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 2nd edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **325**. Springer-Verlag, Berlin, 2005

[3] P. Embid, J. Goodman and A. Majda, Multiple steady states for 1-D transonic flow. *SIAM J. Sci. Stat. Comput.* **5** (1984), 21–41.

[4] N. Fenichel, Persistence and smoothness of invariant manifolds and flows. *Indiana Univ. Math. J.* **21** (1971), 193–226.

[5] P. Hartman, *Ordinary Differential Equations.* John Wiley & Sons, Inc., New York-London-Sydney, 1964

[6] J. M. Hong, C.-H. Hsu and W. Liu, Viscous standing asymptotic states of isentropic compressible flows through a nozzle. *Arch. Ration. Mech. Anal.* (accepted).

[7] J. M. Hong, C.-H. Hsu and W. Liu, Viscous standing asymptotic states of gas flow through a contracting-expanding nozzle. *Preprint.*

[8] J. M. Hong, C.-H. Hsu and W. Liu, Sub-to-super transonic steady states and their linear stabilities for gas flows. *Preprint.*

[9] M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds.* Lecture Notes in Math. **583**, Springer-Verlag, New York, 1976

[10] S.-B. Hsu and T.P. Liu, Nonlinear singular Sturm-Liouville problems and an application to transonic flow through a nozzle. *Comm. Pure Appl. Math.* **43** (1990), 31–36.

[11] P. D. Lax, Hyperbolic system of conservation laws, II. *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[12] H. W. Liepmann and A. Roshko, *Elementary of Gas Dynamics.* GALCIT Aeronautical Series, New York: Wiely, 1957

[13] T. P. Liu, Quasilinear hyperbolic system. *Comm. Math. Phys.* **68** (1979), 141–172.

[14] T. P. Liu, Transonic gas flow in a duct of varying area. *Arch. Ration. Mech. Anal.* **80** (1982), 1–18.

[15] B. Sandstede, Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equation. *SIAM J. Math. Anal.* **29** (1998), 183–207.

[16] D. Serre, *Systems of conservation laws. 1. Hyperbolicity, entropies, shock waves.* Translated from the 1996 French original by I. N. Sneddon. Cambridge University Press, Cambridge, 1999

[17] D. Serre, *Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems.* Translated from the 1996 French original by I. N. Sneddon. Cambridge University Press, Cambridge, 2000

[18] B. Whitham, *Linear and nonlinear waves.* New York, John Wiley, 1974