Poisson-Nernst-Planck systems for narrow tubular-like membrane channels

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Dedicated to Professor Jack K. Hale on the occasion of his 80th birthday

Abstract

We study global asymptotic behavior of Poisson-Nernst-Planck (PNP) systems for flow of two ion species through a narrow tubular-like membrane channel. As the radius of the cross-section of the three-dimensional tubular-like membrane channel approaches zero, a one-dimensional limiting PNP system is derived. This one-dimensional limiting system differs from previously studied one-dimensional PNP systems in that it encodes the defining geometry of the three-dimensional membrane channel. To justify this limiting process, we show that the global attractors of the three-dimensional PNP systems are upper semi-continuous as the radius of the channel tends to zero.

Key words. Poisson-Nernst-Planck system, global attractors, upper semi-continuity.

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1 Introduction

Poisson-Nernst-Planck (PNP) systems serve as basic electro-diffusion equations modeling, for example, ion flow through membrane channels, transport of holes and electrons in semiconductors (see, e.g., [1, 2, 24] and the references therein). PNP systems have been studied under various physically relevant boundary conditions such as the non-flux, homogeneous Dirichlet and homogeneous Neumann boundary conditions. For those types of boundary conditions, in addition to the total charge conservation and the existence of various first integrals, Boltzmann H-functionals or entropy-like functionals are successfully constructed, which, together with the advances of Csiszár-Kullback-type or logarithmic Soblev inequalities, are applied to investigate the asymptotic behavior of the PNP systems and stability of steady-state or self-similar solutions (see, e.g., [3, 4, 6, 7, 8, 17, 25]). In the context of ion flow through membrane channels, it is physically unreasonable to impose the above mentioned boundary conditions on the whole boundary, particularly at the two "ends" of the channels. Instead, non-homogeneous Dirichlet conditions on the two "ends" are typically assumed. PNP systems supplemented with this type of boundary conditions result in quite different dynamical behavior. The total charges within the channels are not conserved and entropy-like functionals are not available in general.

In this work, we consider the PNP system modeling ion flow through narrow tubular-like membrane channels and examine a reduction of the three dimensional PNP system to a one-dimensional limiting PNP system. For simplicity, we consider flow of two ion species, S_1 and S_2, one with positive valence $\alpha_1 > 0$ and the other with negative valence $-\alpha_2 < 0$, passing through a membrane channel with length normalized from $X = 0$ to $X = 1$. Denote the concentrations of S_1 and S_2 at location (X, Y, Z) in the channel and time t by $c_1(t, X, Y, Z)$ and $c_2(t, X, Y, Z)$. Then the electric potential $\Phi(t, X, Y, Z)$ in the channel is governed by the Poisson equation

$$\triangle\Phi = -\lambda(\alpha_1 c_1 - \alpha_2 c_2),$$

where the parameter λ is the Debye number related to the ratio of the Debye length to a characteristic length scale. The flux densities, $\bar{J}_1$ and $\bar{J}_2$, of the two ion species contributed from the concentration gradients of the two ion species and the electric field satisfy the Nernst-Planck equations

$$D_1(\nabla c_1 + \alpha_1 c_1 \nabla\Phi) = -\bar{J}_1 \text{ and } D_2(\nabla c_2 - \alpha_2 c_2 \nabla\Phi) = -\bar{J}_2,$$

and the continuity equations

$$\frac{\partial c_1}{\partial t} + \nabla\bar{J}_1 = 0, \quad \frac{\partial c_2}{\partial t} + \nabla\bar{J}_2 = 0,$$

where D_1 and D_2 are the diffusion constants of ion species S_1 and S_2 relative to the membrane channel. The Poisson-Nernst-Planck system is thus given by

$$
\begin{aligned}
\triangle\Phi =& -\lambda(\alpha_1 c_1 - \alpha_2 c_2),\\
\frac{\partial c_1}{\partial t} =& D_1 \nabla\cdot(\nabla c_1 + \alpha_1 c_1 \nabla\Phi),\\
\frac{\partial c_2}{\partial t} =& D_2 \nabla\cdot(\nabla c_2 - \alpha_2 c_2 \nabla\Phi).
\end{aligned}
\tag{1.1}
$$

PNP systems have been studied by many authors (see, e.g., [1, 2, 3, 4, 6, 7, 8, 12, 13, 14, 17, 19, 24, 25]). Many works have been attributed to a simple one-dimensional version of the PNP system and particularly the steady-state problem (see, e.g., [12, 13, 14, 15, 19]). Consideration of one-dimensional PNP systems is motivated naturally by the fact that membrane channels are *narrow*.

The purpose of this paper is two-fold. First of all, we interpret the *narrowness* of the three-dimensional channel in such a way that we can (mathematically) take the limit of the characteristic radius of the channel to approach zero and derive a reduction of the three-dimensional PNP system; More specifically, starting with the situation that Ω_ε is a revolution domain about its axis (ε is related to the maximal radius of the cross-sections of the channel), we will let $\varepsilon \to 0$ and obtain a one-dimensional limiting PNP system. This limiting process of shrinking a three-dimensional tubular-like domain to a one-dimensional segment is based on the ideas in [10, 11, 20, 21, 22] for thin domains (some discussions of these works will be given in Section 2.1). Differing from the simple one-dimensional version of the PNP system, this one-dimensional limiting PNP system encodes the defining geometry of the three-dimensional channel. Interestingly, the one-dimensional limiting PNP system that we obtained agrees completely with that suggested by Nonner and Eisenberg from physical consideration ([18]). The second purpose is to justify this limiting process. The well-posedness and existence of global attractors for some general evolution systems including the three-dimensional PNP systems have been established by Gajewski and Gröger (see [6, 7] and more discussions in Section 2.2). A key ingredient in their work is the existence of an invariant region for the dynamics of the system. The existence of the global attractor $\mathcal{A}_0$ of the one-dimensional limiting PNP system follows from their results too. Our main result toward a justification of the limiting process is the upper semi-continuity of the global attractors $\mathcal{A}_\varepsilon$ of the three-dimensional PNP systems to $\mathcal{A}_0$ at $\varepsilon = 0$. It is expected that if the one-dimensional limiting system is structurally stable, then its dynamics determines that of three-dimensional system for $\varepsilon > 0$ small. Alone this line, for large Debye number λ, the steady-state problem of the one-dimensional limiting PNP system can be viewed as a singularly perturbed one and has been completely analyzed using the geometric singular perturbation theory as in [5, 15, 16].

The rest of the paper is organized as follows. In Section 2, we give a detailed formulation of our problem. The domain for the three-dimensional PNP system will be specified and, as the domain shrinks to a one-dimensional segment, a one-dimensional limiting PNP system is derived. We then state our result (Theorem 2.2) on the upper semi-continuity of attractors of the three-dimensional PNP system at the limit. A proof of Theorem 2.2 is provided in Section 3. The appendix, Section 4, contains two simple lemmas that are used in the derivation of the one-dimensional limiting PNP system in Section 2.1 and a homogenization of the boundary conditions in Section 3.1.

2 Reformulation of PNP and main results

2.1 Three-dimensional PNP and one-dimensional limits

We start with setting up our problem. The membrane channel considered here is special and will be viewed as a tubular-like domain Ω_ε in $\mathbb{R}^3$ as follows:

$$\Omega_\varepsilon = \{(X,Y,Z) : 0 < X < 1,\ Y^2 + Z^2 < g^2(X,\varepsilon)\},$$

where the function $g \in \mathcal{C}^3$ is the radius of the varying cross-section of the channel. We assume

$$g(X,0) = 0 \ \text{ and } \ g_0(X) = \frac{\partial g}{\partial \varepsilon}(X,0) > 0 \ \text{ for } \ X \in [0,1] \tag{2.1}$$

so that the parameter ε measures the sizes of cross-sections of the membrane channel. For example, one can take $g(X,\varepsilon) = \varepsilon g_0(X)$. For a technical reason (used in Lemma 4.2 in the Appendix), we also assume that

$$\frac{\partial g}{\partial X}(0,\varepsilon) = \frac{\partial g}{\partial X}(1,\varepsilon) = 0.$$

The boundary $\partial\Omega_\varepsilon$ of Ω_ε will be divided into three portions as follows:

$$\begin{aligned}\hat{L}_\varepsilon =& \{(X,Y,Z) \in \partial\Omega_\varepsilon : X = 0\},\\ \hat{R}_\varepsilon =& \{(X,Y,Z) \in \partial\Omega_\varepsilon : X = 1\},\\ \hat{M}_\varepsilon =& \{(X,Y,Z) \in \partial\Omega_\varepsilon : Y^2 + Z^2 = g^2(X,\varepsilon)\}.\end{aligned}$$

Thus, $\hat{L}_\varepsilon$ and $\hat{R}_\varepsilon$ are viewed as the two ends of the channel and $\hat{M}_\varepsilon$ the wall of the channel. The boundary conditions considered in this paper are

$$\begin{gathered}\Phi|_{\hat{L}_\varepsilon} = \phi_0 > 0, \quad \Phi|_{\hat{R}_\varepsilon} = 0, \quad c_k|_{\hat{L}_\varepsilon} = l_k > 0, \quad c_k|_{\hat{R}_\varepsilon} = r_k > 0,\\ \frac{\partial \Phi}{\partial \mathbf{n}}|_{\hat{M}_\varepsilon} = \frac{\partial c_k}{\partial \mathbf{n}}|_{\hat{M}_\varepsilon} = 0,\end{gathered} \tag{2.2}$$

where the constants ϕ_0, l_k and r_k ($k = 1,2$) are the relative electric potential and the concentrations of the two ion species at the two ends, and $\mathbf{n}$ is the outward unit normal vector to $\hat{M}_\varepsilon$. Although the most natural boundary conditions on $\hat{M}_\varepsilon$ would be the non-flux one

$$\left(\frac{\partial c_1}{\partial \mathbf{n}} + \alpha_1 c_1 \frac{\partial \Phi}{\partial \mathbf{n}}\right)|_{\hat{M}_\varepsilon} = \left(\frac{\partial c_2}{\partial \mathbf{n}} - \alpha_2 c_2 \frac{\partial \Phi}{\partial \mathbf{n}}\right)|_{\hat{M}_\varepsilon} = 0,$$

the above homogeneous Neumann conditions on $\hat{M}_\varepsilon$ are reasonable (they are the consequences of the non-flux and zero-outward electric field conditions).

In this paper, we are interested in the limiting behavior of the PNP system when the three-dimensional tubular-like domain Ω_ε shrinks to a one-dimensional interval as $\varepsilon \to 0$. Naturally we expect a one-dimensional limiting PNP system whose global dynamics is comparable with those of PNP systems for $\varepsilon > 0$ small. This important idea has been applied by many researchers in studying the dynamics of equations defined on thin domains. In [11], Hale and Raugel consider a reaction-diffusion equation $(RD)_\varepsilon$ on a thin domain $\Omega_\varepsilon \subset \mathbb{R}^{n+1}$, $n \le 2$, with $\Omega_\varepsilon \to \Omega \subset \mathbb{R}^n$ as $\varepsilon \to 0$. Under some conditions, they derive an appropriate limiting equation $(RD)_0$ on Ω and establish relationships between the solutions of the two systems. Based on uniform estimates of solutions in small ε, they prove the upper semi-continuity of the attractors $\mathcal{A}_\varepsilon$ for $(RD)_\varepsilon$ at $\varepsilon = 0$. Under further conditions, they even establish the topological equivalence between the dynamics defined by $(RD)_\varepsilon$ and $(RD)_0$ for $\varepsilon > 0$ small. The idea is further applied in [10] for a damped hyperbolic equation on a thin domain. In [20, 21, 22], Raugel and Sell study the Navier-Stokes equations $(NS)_\varepsilon$ on a thin three-dimensional domain $\Omega_\varepsilon = Q_2 \times (0, \varepsilon)$, where Q_2 is a bounded domain in $\mathbb{R}^2$. Given $\varepsilon > 0$, they show that there is a subset $B(\varepsilon)$ in $H^1(\Omega_\varepsilon)$ such that for every initial condition in $B(\varepsilon)$, the three-dimensional equation $(NS)_\varepsilon$ has a strong solution in $C([0,\infty), H^1(\Omega_\varepsilon))$. It is interesting that the radius of the set $B(\varepsilon)$ may approach infinity as $\varepsilon \to 0$. They further prove that the dynamical system generated by $(NS)_\varepsilon$ has a local attractor $\mathcal{A}_\varepsilon$ in $H^1(\Omega_\varepsilon)$. This attractor is compact in $H^2(\Omega_\varepsilon)$ and attracts all weak solutions at $t \to \infty$. The upper semi-continuity of $\mathcal{A}_\varepsilon$, as $\varepsilon \to 0$, is also established. Although we are shrinking a three-dimesional domain to a one-dimensional one, the ideas and the procedures in the above mentioned references work for our problem. To be specific, we follow the procedure in [11] to derive a one-dimensional limiting system in this section and prove the upper semi-continuity of attractors in the next section. For a technical purpose, we state a lemma (Lemma 4.1 in the Appendix) that allows us to express differential operators under transformations in a more compact way and, as a result, the expected one-dimensional limiting system is more transparent.

To derive the limiting PNP system, we transfer the ε-dependent domain Ω_ε into a fixed domain $\Omega = [0,1] \times \mathbb{D}$, where $\mathbb{D}$ is the unit disk, by applying the following change of coordinates:

$$x = X,\ y = \frac{Y}{g(X,\varepsilon)},\ z = \frac{Z}{g(X,\varepsilon)}. \tag{2.3}$$

In this way, we can use one phase space, say $H^1(\Omega)$, for the three-dimensional PNP systems corresponding to all $\varepsilon > 0$ and even for the one-dimensional limiting system defined on $[0,1]$ since a function on $[0,1]$ can be naturally identified as a function on Ω.

In the sequel, we denote by L, R and M, respectively, the boundaries of Ω corresponding to $\hat{L}_\varepsilon$, $\hat{R}_\varepsilon$ and $\hat{M}_\varepsilon$ under the transformation. Let J denote the Jacobian matrix of the change of coordinates. Then,

$$J = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \frac{1}{g^2}\begin{pmatrix} g^2 & 0 & 0 \\ -gg_x y & g & 0 \\ -gg_x z & 0 & g \end{pmatrix}, \quad J^{-1} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ g_x y & g & 0 \\ g_x z & 0 & g \end{pmatrix}$$

with $\det(J^{-1}) = g^2(x,\varepsilon)$, and

$$JJ^\tau = \frac{1}{g^4}\begin{pmatrix} g^4 & -g^3 g_x y & -g^3 g_x z \\ -g^3 g_x y & g^2 + g^2 g_x^2 y^2 & g^2 g_x^2 yz \\ -g^3 g_x z & g^2 g_x^2 yz & g^2 + g^2 g_x^2 z^2 \end{pmatrix}.$$

It can be checked that the change of variables in (2.3) with $p = (X,Y,Z)$ and $q = (x,y,z)$ satisfies

$$\sum_{j=1}^n \frac{\partial}{\partial q_j}\left(d(q)\frac{\partial q_j}{\partial p_i}\right) = 0.$$

Therefore, applying Lemma 4.1 in the Appendix, system (1.1) can be rewritten, in terms of (x,y,z), as follows:

$$\begin{aligned} &\frac{1}{g^2}\nabla\cdot\left(g^2 JJ^\tau \nabla\Phi\right) = -\lambda(\alpha_1 c_1 - \alpha_2 c_2), \\ &\frac{\partial c_1}{\partial t} = \frac{D_1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla c_1 + \alpha_1 c_1 g^2 JJ^\tau \nabla\Phi), \\ &\frac{\partial c_2}{\partial t} = \frac{D_2}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla c_2 - \alpha_2 c_2 g^2 JJ^\tau \nabla\Phi), \end{aligned} \tag{2.4}$$

with the boundary conditions

$$\begin{aligned} \Phi|_L = \phi_0, \quad \Phi|_R = 0, \quad c_k|_L = l_k, \quad c_k|_R = r_k, \\ \langle\nabla\Phi, JJ^\tau\nu\rangle|_M = \langle\nabla c_k, JJ^\tau\nu\rangle|_M = 0, \end{aligned} \tag{2.5}$$

where $k = 1,2$ and ν is the outward unit normal vector to M.

By inspecting the structural dependence of JJ^τ on ε, we expect the one-dimensional limiting PNP system to be

$$\begin{aligned} &\frac{1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}\Phi\right) = -\lambda(\alpha_1 c_1 - \alpha_2 c_2), \\ &\frac{\partial c_1}{\partial t} = \frac{D_1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}c_1 + \alpha_1 c_1 g_0^2\frac{\partial}{\partial x}\Phi\right), \\ &\frac{\partial c_2}{\partial t} = \frac{D_2}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}c_2 - \alpha_2 c_2 g_0^2\frac{\partial}{\partial x}\Phi\right), \end{aligned} \tag{2.6}$$

on the interval $(0,1)$ with the boundary conditions

$$\Phi(t,0) = \phi_0, \quad \Phi(t,1) = 0, \quad c_k(t,0) = l_k, \quad c_k(t,1) = r_k, \tag{2.7}$$

where $g_0(x)$ is defined in (2.1).

Note that Φ can be solved in terms of c_1 and c_2 for both the three-dimensional and one-dimensional limiting PNP systems. In the sequel, we will mainly treat the systems for unknowns c_1 and c_2; in particular, the phase space for both systems consists of functions (c_1, c_2). Properties on Φ follow easily from those of (c_1, c_2).

2.2 Invariant region and existence of attractors

In [7], Gajewski and Gröger study semiconductor equations describing the transport of mobile carriers, electrons and holes, in semiconductor devices proposed by van Roosbroeck ([23]). In their model equations, in addition to other generalities, they treat both Boltzmann statistics and Fermi-Dirac statistics for the dependence of the carrier densities on the corresponding chemical potentials. It was shown in [7] that the semiconductor equation (for both Boltzmann statistics and Fermi-Dirac statistics) has a global attractor which is a compact subset and attracts all solutions with respect to the norm topology of $H^1 \times H^1$. The three-dimensional PNP systems (1.1) and (2.2) considered in this work correspond to the semiconductor equations for the Boltzmann statistics case, and hence, for any $\varepsilon > 0$, there exists a compact global attractor $\mathcal{A}_\varepsilon$. A key ingredient for their result is the existence of an invariant region discovered in [8, 9, 25]. For the three-dimensional PNP system (1.1) and (2.2), the result on the existence of an invariant region is recorded below.

Proposition 2.1. *Let M be a positive constant with $M \geq \max\{\alpha_1 l_1, \alpha_1 r_1, \alpha_2 l_2, \alpha_2 r_2\}$. Then*

$$\tilde{\Sigma} = \{(c_1, c_2) \in H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) : \ 0 \leq \alpha_1 c_1 \leq M, \ 0 \leq \alpha_2 c_2 \leq M\}$$

is positively invariant for the PNP system. More precisely, if the initial datum $(c_1(0), c_2(0)) \in \tilde{\Sigma}$ and (c_1, c_2) is the solution of the PNP system, then $(c_1(t), c_2(t)) \in \tilde{\Sigma}$ for all $t \geq 0$.

One can check that the following region is positively invariant for the one-dimensional problem (2.6)-(2.7)

$$\tilde{\Sigma}_0 = \{(c_1, c_2) \in H^1(0,1) \times H^1(0,1) : \ 0 \leq \alpha_1 c_1 \leq M, \ 0 \leq \alpha_2 c_2 \leq M\},$$

where M is the constant in Proposition 2.1. Applying the results in [7], we also conclude that problem (2.6)-(2.7) has a global attractor $\mathcal{A}_0$ in $\tilde{\Sigma}_0 \cap H^1(0,1) \times H^1(0,1)$.

We remark that, for PNP systems with three or more types of ions, the above invariant principle is not available. It is not clear to us whether or not a similar principle still holds in this case. PNP systems with more than two types of ions are worth further studying.

2.3 Main result

Our main result claims that the global attractors $\mathcal{A}_\varepsilon$ of the three-dimensional PNP systems are upper semi-continuous to the global attractor $\mathcal{A}_0$ of the one-dimensional limiting system as $\varepsilon \to 0$, which partially justify the limiting process. More precisely, we have

Theorem 2.2. *The global attractors $\mathcal{A}_\varepsilon$ of the three-dimensional PNP systems are upper semi-continuous at $\varepsilon = 0$, that is, for any $\eta > 0$, there exists a positive number $\varepsilon_1 = \varepsilon_1(\eta)$ such that for all $0 < \varepsilon \le \varepsilon_1$ and all $w \in \mathcal{A}_\varepsilon$,*

$$\mathrm{dist}_{X_\varepsilon}(w, \mathcal{A}_0) \le \eta,$$

where $X_\varepsilon = \left\{ w : \ \|w\|^2_{X_\varepsilon} = \|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} + \frac{1}{\varepsilon^2}\|w_y\|^2_{L^2} + \frac{1}{\varepsilon^2}\|w_z\|^2_{L^2} < \infty \right\}$.

Remark 2.1. For large Debye number λ, the steady-state problem of (2.6) can be viewed as a singular perturbation problem and, indeed, the existence of steady solutions of the system was proved in the first version of this paper by using the geometric singular perturbation theory as discussed in [15]. Since the completion of the present paper, the steady-state problem for more general limiting PNP systems has been analyzed in [5] and [16]. Consequently, we will not present the existence of steady solutions for system (2.6) in this updated version. The reader is referred to [5] and [16] for detailed results on steady-state problems for the cases of *two* and *multiple* ion species with permanent charges, respectively.

3 Uniform estimates and a proof of Theorem 2.2

3.1 Homogenization of boundary conditions

In this section, we convert the non-homogeneous Dirichlet boundary conditions on $L \cup R$ in (2.5) to homogeneous ones, while keeping the homogeneous Neumann boundary conditions on M.

Let $L_k^0(X)$, for $k = 1, 2, 3$, be the linear functions satisfying $L_k^0(0) = l_k$, $L_k^0(1) = r_k$ for $k = 1, 2$, $L_3^0(0) = \phi_0$ and $L_3^0(1) = 0$. Lemma 4.2 in the Appendix guarantees the existence of functions $L_k(X, Y, Z, \varepsilon)$ for $k = 1, 2, 3$ such that for each $\varepsilon > 0$ and $Y^2 + Z^2 < g^2(X, \varepsilon)$, $L_k(X, 0, 0, \varepsilon) = L_k^0$, $L_k(0, Y, Z, \varepsilon) = L_k^0(0)$, $L_k(1, Y, Z, \varepsilon) = L_k^0(1)$, and $\frac{\partial}{\partial \mathbf{n}} L_k(X, Y, Z, \varepsilon) = 0$ when $(X, Y, Z) \in \hat{M}_\varepsilon$. For each $\varepsilon > 0$ and $k = 1, 2, 3$, introduce the functions L_k^ε in terms of variables x, y and z:

$$L_k^\varepsilon(x, y, z) = L_k(X, Y, Z, \varepsilon) = L_k(x, g(x, \varepsilon)y, g(x, \varepsilon)z, \varepsilon). \tag{3.1}$$

Set

$$
\begin{aligned}
u(x,y,z) =& L_1^\varepsilon(x,y,z) - c_1(x,y,z),\\
v(x,y,z) =& L_2^\varepsilon(x,y,z) - c_2(x,y,z),\\
\phi(x,y,z) =& L_3^\varepsilon(x,y,z) - \Phi(x,y,z).
\end{aligned}
$$

Then, problem (2.4)-(2.5) is transformed into

$$
\begin{aligned}
&\frac{1}{g^2}\nabla\cdot\left(g^2JJ^\tau\nabla(\phi-L_3^\varepsilon)\right) = -\lambda(\alpha_1(u-L_1^\varepsilon)-\alpha_2(v-L_2^\varepsilon)),\\
&\frac{\partial u}{\partial t} = \frac{D_1}{g^2}\nabla\cdot\left(g^2JJ^\tau\nabla(u-L_1^\varepsilon)-\alpha_1(u-L_1^\varepsilon(x))g^2JJ^\tau\nabla(\phi-L_3^\varepsilon)\right),\\
&\frac{\partial v}{\partial t} = \frac{D_2}{g^2}\nabla\cdot\left(g^2JJ^\tau\nabla(v-L_2^\varepsilon)+\alpha_2(v-L_2^\varepsilon(x))g^2JJ^\tau\nabla(\phi-L_3^\varepsilon)\right),
\end{aligned}
\tag{3.2}
$$

with the homogeneous boundary conditions:

$$
\begin{aligned}
\phi|_{L\cup R} =& u|_{L\cup R} = v|_{L\cup R} = 0,\\
\langle\nabla\phi, JJ^\tau\nu\rangle|_M =& \langle\nabla u, JJ^\tau\nu\rangle|_M = \langle\nabla v, JJ^\tau\nu\rangle|_M = 0.
\end{aligned}
\tag{3.3}
$$

System (3.2) is supplemented with the initial conditions:

$$
u(0) = u_0, \quad v(0) = v_0. \tag{3.4}
$$

Introduce the subspace $H_D^1(\Omega)$ of $H^1(\Omega)$:

$$
H_D^1(\Omega) = \{u \in H^1(\Omega) : \ u|_{L\cup R} = 0\}.
$$

Let M be the constant in Proposition 2.1 and let Σ_ε be the subset of $H_D^1(\Omega)\times H_D^1(\Omega)$ given by

$$
\Sigma_\varepsilon = \{(u,v)\in H_D^1(\Omega)\times H_D^1(\Omega) : \ \alpha_1L_1^\varepsilon - M \le \alpha_1 u \le \alpha_1L_1^\varepsilon, \ \alpha_2L_2^\varepsilon - M \le \alpha_2 v \le \alpha_2 L_2^\varepsilon\}. \tag{3.5}
$$

It follows from Proposition 2.1 that, if $(u_0,v_0)\in\Sigma_\varepsilon$, then $(u(t),v(t))\in\Sigma_\varepsilon$ for every $t\ge 0$. Throughout this paper, for every $\varepsilon>0$, we denote by $S^\varepsilon(t)_{t\ge0}$ the solution operator associated with problem (3.2)-(3.4). We will use the same symbol $\mathcal{A}_\varepsilon$ to denote the global attractors of $S^\varepsilon(t)_{t\ge0}$ and that of problem (2.4)-(2.5) when no confusion arises.

The corresponding one-dimensional limiting system (2.6) is transformed into

$$
\begin{aligned}
&\frac{1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}\left(\phi-L_3^0\right)\right) = -\lambda\left(\alpha_1(u-L_1^0)-\alpha_2(v-L_2^0)\right),\\
&\frac{\partial u}{\partial t} = \frac{D_1}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}(u-L_1^0)-\alpha_1(u-L_1^0)g_0^2\frac{\partial}{\partial x}(\phi-L_3^0)\right),\\
&\frac{\partial v}{\partial t} = \frac{D_2}{g_0^2}\frac{\partial}{\partial x}\left(g_0^2\frac{\partial}{\partial x}(v-L_2^0)+\alpha_2(v-L_2^0)g_0^2\frac{\partial}{\partial x}(\phi-L_3^0)\right),
\end{aligned}
\tag{3.6}
$$

with the homogeneous Dirichlet boundary conditions

$$\phi = u = v = 0, \quad x = 0, 1, \tag{3.7}$$

and the initial conditions

$$u(0) = u_0, \quad \text{and } v(0) = v_0. \tag{3.8}$$

The following result can be proved in the same way as that for Proposition 2.1.

Proposition 3.1. *Let M be a positive constant with $M \geq \max\{\alpha_1 l_1, \alpha_1 r_1, \alpha_2 l_2, \alpha_2 r_2\}$. Then*

$$\Sigma_0 = \{(u, v) \in H_0^1(0,1) \times H_0^1(0,1) : \ \alpha_1 L_1^0 - M \leq \alpha_1 u \leq \alpha_1 L_1^0, \ \alpha_2 L_2^0 - M \leq \alpha_2 v \leq \alpha_2 L_2^0\} \tag{3.9}$$

is positively invariant for the one-dimensional PNP system (3.6)-(3.8).

Similar to system (2.6)-(2.7), problem (3.6)-(3.8) is well-posed in Σ_0, that is, for each $(u_0, v_0) \in \Sigma_0$, there exists a unique solution (u, v) for problem (3.6)-(3.8) which is defined for all $t \geq 0$ and $(u, v) \in \mathcal{C}([0, \infty), \Sigma_0)$. Further, the solutions are continuous in initial data with respect to the topology of $H_0^1(0,1) \times H_0^1(0,1)$. Therefore, there is a continuous dynamical system $S^0(t)_{t \geq 0}$ associated with problem (3.6)-(3.8) such that for each $t \geq 0$ and $(u_0, v_0) \in \Sigma_0$, $S^0(t)(u_0, v_0) = (u(t), v(t))$ is the solution of problem (3.6)-(3.8). When no confusion arises, we use the same symbol $\mathcal{A}_0$ to denote the global attractors of $S^0(t)_{t \geq 0}$ and problem (2.6)-(2.7).

3.2 Uniform estimates of global attractors

In this section, we derive uniform estimates of the global attractors $\mathcal{A}_\varepsilon$ in ε which are necessary for establishing the upper semi-continuity of $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$. In what follows, we reformulate problem (3.2)-(3.4) as an abstract differential equation in $H_D^1(\Omega) \times H_D^1(\Omega)$.

Given $\varepsilon > 0$, define an inner product $(\cdot, \cdot)_{H_\varepsilon}$ on $L^2(\Omega)$ by

$$(v, w)_{H_\varepsilon} = \int_\Omega \frac{g^2}{\varepsilon^2} vw \, dx \, dy \, dz,$$

and a bilinear form $a_\varepsilon(\cdot, \cdot)$ on $\left(H_D^1(\Omega)\right)^2$ by

$$a_\varepsilon(w_1, w_2) = (J^\tau \nabla w_1, J^\tau \nabla w_2)_{H_\varepsilon} = \int_\Omega \frac{g^2}{\varepsilon^2} J^\tau \nabla w_1 \cdot J^\tau \nabla w_2 \, dx \, dy \, dz.$$

In the sequel, we denote $\|w\|_p$ the standard norm of w for $w \in L^p(\Omega)$ or $w \in L^p([0,1])$, $\|w\|_{H^s}$ the standard norm of w for $w \in H^s(\Omega)$ or $w \in H^s([0,1])$. Also, denote H_ε the space $L^2(\Omega)$ with the inner product $(\cdot, \cdot)_{H_\varepsilon}$, and X_ε the space $H_D^1(\Omega)$ with the norm

$$\|w\|_{X_\varepsilon} = \left(\|\nabla w\|_2^2 + \frac{1}{\varepsilon^2} \|w_y\|_2^2 + \frac{1}{\varepsilon^2} \|w_z\|_2^2 \right)^{1/2}.$$

Since Poincare inequality holds in $H^1_D(\Omega)$, the norm $\|w\|_{X_\varepsilon}$ for $w \in H^1_D(\Omega)$ is equivalent to the norm given by

$$\left(\|w\|^2_{H^1} + \frac{1}{\varepsilon^2}\|w_y\|^2_2 + \frac{1}{\varepsilon^2}\|w_z\|^2_2 \right)^{1/2}.$$

Due to assumption (2.1), there exist positive constants C_1, C_2, C_3 (independent of ε) and ε_1 such that, for all $0 < \varepsilon \le \varepsilon_1$ and $x \in (0,1)$,

$$\frac{|g_x|}{g} \le C_1, \quad C_2 \le \frac{g}{\varepsilon} \le C_3. \tag{3.10}$$

Consequently, $\sqrt{a_\varepsilon(w,w)}$ is equivalent to the norm $\|w\|_{X_\varepsilon}$, that is,

$$C_4\|w\|^2_{X_\varepsilon} \le a_\varepsilon(w,w) \le C_5\|w\|^2_{X_\varepsilon} \tag{3.11}$$

for some constants C_4 and C_5 (independent of ε). It follows from (3.11) that for each $\varepsilon > 0$, the triple $\{H^1_D(\Omega), H_\varepsilon, a_\varepsilon(\cdot,\cdot)\}$ defines a unique unbounded operator $\mathcal{L}_\varepsilon$ on $H^1_D(\Omega)$ with domain $D(\mathcal{L}_\varepsilon)$ in the following way: an element $u \in H^1_D(\Omega)$ belongs to $D(\mathcal{L}_\varepsilon)$ if $a_\varepsilon(u,v)$ is continuous in $v \in H^1_D(\Omega)$ for the topology induced from H_ε and $(\mathcal{L}_\varepsilon u, v)_{H_\varepsilon} = a_\varepsilon(u,v)$ for $(u,v) \in D(\mathcal{L}_\varepsilon) \times H^1_D(\Omega)$. In fact,

$$D(\mathcal{L}_\varepsilon) = \{u \in H^1_D(\Omega) : \ \mathcal{L}_\varepsilon u \in H_\varepsilon\},$$

and for every $u \in D(\mathcal{L}_\varepsilon)$,

$$\mathcal{L}_\varepsilon u = -\frac{1}{g^2}\nabla \cdot \left(g^2 J J^\tau \nabla u\right).$$

Since the operator $\mathcal{L}_\varepsilon$ is self-adjoint on H_ε and positive, the fractional power $\mathcal{L}_\varepsilon^{1/2}$ is well-defined with domain $D(\mathcal{L}_\varepsilon^{1/2}) = H^1_D(\Omega)$, and for $u \in H^1_D(\Omega)$,

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|^2_{H_\varepsilon} = a_\varepsilon(u,u).$$

In view of (3.11) there exist C_6 and C_7 such that

$$C_6\|u\|_{X_\varepsilon} \le \|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon} \le C_7\|u\|_{X_\varepsilon}. \tag{3.12}$$

With the above notations, system (3.2) can be rewritten as

$$\begin{aligned} \mathcal{L}_\varepsilon \phi &= \lambda\alpha_1(u - L^\varepsilon_1) - \lambda\alpha_2(v - L^\varepsilon_2) - \frac{1}{g^2}\nabla \cdot \left(g^2 J J^\tau \nabla L^\varepsilon_3\right), \\ \frac{\partial u}{\partial t} + D_1\mathcal{L}_\varepsilon u &= -\frac{D_1}{g^2}\nabla \cdot \left(g^2 J J^\tau \nabla L^\varepsilon_1 + \alpha_1(u - L^\varepsilon_1) g^2 J J^\tau \nabla(\phi - L^\varepsilon_3)\right), \\ \frac{\partial v}{\partial t} + D_2\mathcal{L}_\varepsilon v &= -\frac{D_2}{g^2}\nabla \cdot \left(g^2 J J^\tau \nabla L^\varepsilon_2 - \alpha_2(v - L^\varepsilon_2) g^2 J J^\tau \nabla(\phi - L^\varepsilon_3)\right). \end{aligned} \tag{3.13}$$

By the construction of functions L_k^ε $(k = 1, 2, 3)$, there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \le \varepsilon_1$, the following uniform bounds in ε hold:

$$\|L_k^\varepsilon\|_\infty + \|L_k^\varepsilon\|_{H_\varepsilon} + \|J^\tau \nabla L_k^\varepsilon\|_{H_\varepsilon} + \|JJ^\tau \nabla L_k^\varepsilon\|_{H_\varepsilon} + \|\frac{1}{g^2}\nabla \cdot \left(g^2 JJ^\tau \nabla L_k^\varepsilon\right)\|_{H_\varepsilon} \le C, \tag{3.14}$$

where C is independent of ε. Then it follows from the positive invariance of Σ_ε that there exists a constant C (independent of ε) such that for any initial datum $(u_0, v_0) \in \Sigma_\varepsilon$, the solution (u, v) of problem (3.2)-(3.4) satisfies, for all $t \ge 0$:

$$\|u(t)\|_\infty + \|v(t)\|_\infty \le C \quad \text{and} \quad \|u(t)\|_{H_\varepsilon} + \|v(t)\|_{H_\varepsilon} \le C. \tag{3.15}$$

Next, we start to derive uniform estimates of solutions in ε in the space $H_D^1(\Omega) \times H_D^1(\Omega)$.

Lemma 3.2. *There exist a constant C (independent of ε) and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \le \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$, the solution (u, v) of problem (3.2)-(3.4) satisfies, for all $t \ge 0$:*

$$\int_t^{t+1} \left(\|u(t)\|_{X_\varepsilon} + \|v(t)\|_{X_\varepsilon}\right) dt \le C.$$

Proof. Taking the inner product of the first equation in (3.13) with ϕ in H_ε, we find that

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon}^2 = \lambda\alpha_1(u - L_1^\varepsilon, \phi)_{H_\varepsilon} - \lambda\alpha_2(v - L_2^\varepsilon, \phi)_{H_\varepsilon} + (J^\tau \nabla L_3^\varepsilon, J^\tau \nabla\phi)_{H_\varepsilon}.$$

By (3.14) and (3.15) we have

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon}^2 \le& \lambda\alpha_1(\|u\|_{H_\varepsilon} + \|L_1^\varepsilon\|_{H_\varepsilon})\|\phi\|_{H_\varepsilon} + \lambda\alpha_2(\|v\|_{H_\varepsilon} + \|L_2^\varepsilon\|_{H_\varepsilon})\|\phi\|_{H_\varepsilon} + \|J^\tau \nabla L_3^\varepsilon\|_{H_\varepsilon}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon} \\ \le& C\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon} \le \frac{1}{2}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon}^2 + \frac{1}{2}C^2,\end{aligned}$$

which implies that

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon} \le C. \tag{3.16}$$

Now, taking the inner product of the second equation in (3.13) with u in H_ε, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H_\varepsilon}^2 + D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon}^2 = D_1\left(J^\tau \nabla L_1^\varepsilon, J^\tau \nabla u\right)_{H_\varepsilon} + D_1\alpha_1\left((u - L_1^\varepsilon)J^\tau \nabla(\phi - L_3^\varepsilon), J^\tau \nabla u\right)_{H_\varepsilon}.$$

It follows from (3.14)-(3.16) that the right-hand side of the above is bounded by

$$\begin{aligned}C_1\|J^\tau \nabla L_1^\varepsilon\|_{H_\varepsilon}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon} +& D_1\alpha_1(\|u\|_\infty + \|L_1^\varepsilon\|_\infty)(\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon} + \|J^\tau \nabla L_3^\varepsilon\|_{H_\varepsilon})\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon} \\ \le& C\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon} \le \frac{1}{2}D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon}^2 + C_1.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u\|_{H_\varepsilon}^2 + D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon}^2 \le C_2. \tag{3.17}$$

Similarly,

$$\frac{d}{dt}\|v\|^2_{H_\varepsilon} + D_2\|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|^2_{H_\varepsilon} \le C_3. \tag{3.18}$$

Hence, for all $t \ge 0$:

$$\frac{d}{dt}\left(\|u\|^2_{H_\varepsilon} + \|v\|^2_{H_\varepsilon}\right) + C_4\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|^2_{H_\varepsilon} + \|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|^2_{H_\varepsilon}\right) \le C_2 + C_3,$$

which, along (3.12) and (3.15), implies Lemma 3.2. □

Lemma 3.3. *There exist positive constants ε_1 and C such that for any $0 < \varepsilon \le \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$, the solution (u, v) of problem (3.2)-(3.4) satisfies, for all $t \ge 1$:*

$$\|\mathcal{L}_\varepsilon \phi(t)\|_{X_\varepsilon} + \|u(t)\|_{X_\varepsilon} + \|v(t)\|_{X_\varepsilon} \le C.$$

Proof. By (3.14), (3.15) and the first equation in (3.13) we get

$$\|\mathcal{L}_\varepsilon \phi\|_{H_\varepsilon} \le C\left(\|u\|_{H_\varepsilon} + \|v\|_{H_\varepsilon} + \|L_1^\varepsilon\|_{H_\varepsilon} + \|L_2^\varepsilon\|_{H_\varepsilon} + \|\frac{1}{g^2}\nabla\cdot\left(g^2 JJ^\tau \nabla L_3^\varepsilon\right)\|_{H_\varepsilon}\right) \le C. \tag{3.19}$$

Taking the inner product of the second equation in (3.13) with $\mathcal{L}_\varepsilon u$ in H_ε, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|^2_{H_\varepsilon} + D_1\|\mathcal{L}_\varepsilon u\|^2_{H_\varepsilon} = &-\left(\frac{D_1}{g^2}\nabla\cdot(g^2 JJ^\tau \nabla L_1^\varepsilon), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon} \\ &-\left(\frac{D_1\alpha_1}{g^2}\nabla\cdot((u - L_1^\varepsilon) g^2 JJ^\tau \nabla(\phi - L_3^\varepsilon)), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon}.\end{aligned} \tag{3.20}$$

By (3.14), the first term on the right-hand side of (3.20) is bounded by

$$\left|\left(\frac{D_1}{g^2}\nabla\cdot(g^2 JJ^\tau \nabla L_1^\varepsilon), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon}\right| \le D_1\|\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau \nabla L_1^\varepsilon)\|_{H_\varepsilon}\|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \le \frac{1}{4}D_1\|\mathcal{L}_\varepsilon u\|^2_{H_\varepsilon} + C. \tag{3.21}$$

For the second term on the right-hand side of (3.20), we have

$$\begin{aligned}&-\left(\frac{D_1\alpha_1}{g^2}\nabla\cdot((u - L_1^\varepsilon) g^2 JJ^\tau \nabla(\phi - L_3^\varepsilon)), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon} \\ &= -D_1\alpha_1\left(\nabla(u - L_1^\varepsilon)\cdot JJ^\tau \nabla(\phi - L_3^\varepsilon), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon} \\ &\quad - D_1\alpha_1\left((u - L_1^\varepsilon)\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau \nabla(\phi - L_3^\varepsilon)), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon}.\end{aligned} \tag{3.22}$$

Using (3.14) and (3.19), the first term on the right-hand side of (3.22) is bounded by

$$
\begin{aligned}
& D_1\alpha_1 |\left(\nabla(u-L_1^\varepsilon)\cdot JJ^\tau\nabla(\phi-L_3^\varepsilon), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon}| \\
& \le D_1\alpha_1 \|\nabla(u-L_1^\varepsilon)\|_3 \|\frac{g^2}{\varepsilon^2} JJ^\tau\nabla(\phi-L_3^\varepsilon)\|_6 \|\mathcal{L}_\varepsilon u\|_2 \\
& \le C\|\nabla(u-L_1^\varepsilon)\|_2^{\frac{1}{2}} \|\nabla(u-L_1^\varepsilon)\|_{H^1}^{\frac{1}{2}} \|\frac{g^2}{\varepsilon^2} JJ^\tau\nabla(\phi-L_3^\varepsilon)\|_{H^1} \|\mathcal{L}_\varepsilon u\|_2 \\
& \le \left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon} + \|J^\tau\nabla L_1^\varepsilon\|_{H_\varepsilon}\right)^{\frac{1}{2}} \left(\|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} + \|J^\tau\nabla L_1^\varepsilon\|_{H_\varepsilon} + \|\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla L_1^\varepsilon)\|_{H_\varepsilon}\right)^{\frac{1}{2}} \\
& \times \left(\|\mathcal{L}_\varepsilon\phi\|_{H_\varepsilon} + \|JJ^\tau\nabla L_3^\varepsilon\|_{H_\varepsilon} + \|\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla L_3^\varepsilon)\|_{H_\varepsilon}\right) \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \\
& \le C\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon} + C\right)^{\frac{1}{2}} \left(\|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} + C\right)^{\frac{1}{2}} \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \\
& \le \frac{1}{8} D_1 \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon}^2 + C\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + C. \qquad (3.23)
\end{aligned}
$$

The second term on the right-hand side of (3.22) can be estimated as

$$
\begin{aligned}
& D_1\alpha_1 \left| \left((u-L_1^\varepsilon)\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla(\phi-L_3^\varepsilon)), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon} \right| \\
& \le D_1\alpha_1 \left(\|u\|_\infty + \|L_1^\varepsilon\|_\infty\right) \|\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla(\phi-L_3^\varepsilon))\|_{H_\varepsilon} \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \\
& \le D_1\alpha_1 \left(\|u\|_\infty + \|L_1^\varepsilon\|_\infty\right) \left(\|\mathcal{L}_\varepsilon\phi\|_{H_\varepsilon} + \|\frac{1}{g^2}\nabla\cdot(g^2 JJ^\tau\nabla L_3^\varepsilon)\|_{H_\varepsilon}\right) \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \\
& \le C\|\mathcal{L}_\varepsilon u\|_{H_\varepsilon} \le \frac{1}{8} D_1 \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon}^2 + C. \qquad (3.24)
\end{aligned}
$$

Combining the estimates (3.22)-(3.24), we obtain

$$
|\left(\frac{D_1\alpha_1}{g^2}\nabla\cdot((u-L_1^\varepsilon)g^2 JJ^\tau\nabla(\phi-L_3^\varepsilon)), \mathcal{L}_\varepsilon u\right)_{H_\varepsilon}| \le \frac{1}{4} D_1 \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon}^2 + C\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + C. \qquad (3.25)
$$

It follows from (3.20), (3.21) and (3.25) that, for all $t \ge 0$,

$$
\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + D_1 \|\mathcal{L}_\varepsilon u\|_{H_\varepsilon}^2 \le C_1 \|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + C_2. \qquad (3.26)
$$

Similarly, for all $t \ge 0$,

$$
\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|_{H_\varepsilon}^2 + D_2 \|\mathcal{L}_\varepsilon v\|_{H_\varepsilon}^2 \le C_1 \|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|_{H_\varepsilon}^2 + C_2. \qquad (3.27)
$$

Hence, we have, for all $t \ge 0$,

$$
\frac{d}{dt}\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + \|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|_{H_\varepsilon}^2\right) + C_3\left(\|\mathcal{L}_\varepsilon u\|_{H_\varepsilon}^2 + \|\mathcal{L}_\varepsilon v\|_{H_\varepsilon}^2\right) \le C_1\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u\|_{H_\varepsilon}^2 + \|\mathcal{L}_\varepsilon^{\frac{1}{2}} v\|_{H_\varepsilon}^2\right) + C_2, \quad (3.28)
$$

which, along with Lemma 3.2 and the uniform Gronwall's lemma, implies that, for all $t \geq 1$,

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}} u(t)\|_{H_\varepsilon}^2 + \|\mathcal{L}_\varepsilon^{\frac{1}{2}} v(t)\|_{H_\varepsilon}^2 \leq C.$$

The above estimate and the first equation in (3.13) conclude the proof. $\square$

Applying Gronwall's lemma to (3.28) for $t \in (0,1)$, then by Lemma 3.3 and the first equation in (3.13) we find that there exists $\varepsilon_1 > 0$ such that, for any $R > 0$, there exists K depending on R such that for any $0 < \varepsilon \leq \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$ with $\|(u_0, v_0)\|_{X_\varepsilon \times X_\varepsilon} \leq R$, the following holds:

$$\|\mathcal{L}_\varepsilon \phi(t)\|_{X_\varepsilon} + \|u(t)\|_{X_\varepsilon} + \|v(t)\|_{X_\varepsilon} \leq K, \quad \text{for } t \geq 0. \tag{3.29}$$

An immediate consequence of Lemma 3.3 also shows that all the global attractors $\mathcal{A}_\varepsilon$ are uniformly bounded in ε in the space $H_D^1(\Omega) \times H_D^1(\Omega)$, that is, the following statement is true.

Proposition 3.4. *There exist positive constants ε_1 and C such that for all $0 < \varepsilon \leq \varepsilon_1$ and $(u, v) \in \mathcal{A}_\varepsilon$, the following holds:*

$$\|(u, v)\|_{X_\varepsilon \times X_\varepsilon} \leq C.$$

The following is an analogue of Lemma 3.3 for the limiting system (3.6)-(3.8).

Lemma 3.5. *There exists $C > 0$ such that for any $(u_0, v_0) \in \Sigma_0$, the solution (u, v) of problem (3.6)-(3.8) satisfies, for all $t \geq 1$:*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq C.$$

In addition, there exists K depending on R when $\|(u_0, v_0)\|_{H^1 \times H^1} \leq R$ such that for all $t \geq 0$:

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq K.$$

Next, we establish estimates on time derivatives of solutions for both the three-dimensional system and the one-dimensional limiting system.

Lemma 3.6. *There exists $\varepsilon_1 > 0$ such that for any $R > 0$, there exists K depending only on R such that for any $0 < \varepsilon \leq \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$ with $\|(u_0, v_0)\|_{X_\varepsilon \times X_\varepsilon} \leq R$, the solution (u, v) of problem (3.2)-(3.4) satisfies*

$$t^2 \left(\|\mathcal{L}_\varepsilon \frac{\partial \phi}{\partial t}\|_{H_\varepsilon}^2 + \|\frac{\partial u}{\partial t}\|_{H_\varepsilon}^2 + \|\frac{\partial v}{\partial t}\|_{H_\varepsilon}^2 \right) + \int_0^t s^2 \left(\|\frac{\partial u}{\partial s}\|_{X_\varepsilon}^2 + \|\frac{\partial v}{\partial s}\|_{X_\varepsilon}^2 \right) ds \leq K e^{Kt}, \quad t \geq 0.$$

Proof. Denote by

$$\tilde{\phi} = \frac{\partial \phi}{\partial t}, \quad \tilde{u} = \frac{\partial u}{\partial t}, \quad \tilde{v} = \frac{\partial v}{\partial t}.$$

Differentiating (3.13) with respect to t, we get

$$\begin{aligned}
\mathcal{L}_\varepsilon \tilde{\phi} &= \lambda\alpha_1\tilde{u} - \lambda\alpha_2\tilde{v},\\
\frac{\partial \tilde{u}}{\partial t} + D_1\mathcal{L}_\varepsilon\tilde{u} &= -\frac{D_1}{g^2}\nabla\cdot\Big(\alpha_1\tilde{u}g^2JJ^\tau\nabla(\phi - L_3^\varepsilon) + \alpha_1(u - L_1^\varepsilon)g^2JJ^\tau\nabla\tilde{\phi}\Big),\\
\frac{\partial \tilde{v}}{\partial t} + D_2\mathcal{L}_\varepsilon\tilde{v} &= -\frac{D_2}{g^2}\nabla\cdot\Big(\alpha_2\tilde{v}g^2JJ^\tau\nabla(\phi - L_3^\varepsilon) + \alpha_2(v - L_2^\varepsilon)g^2JJ^\tau\nabla\tilde{\phi}\Big).
\end{aligned}$$

From the above system, one derives

$$\begin{aligned}
\mathcal{L}_\varepsilon(t\tilde{\phi}) &= \lambda\alpha_1 t\tilde{u} - \lambda\alpha_2 t\tilde{v},\\
\frac{\partial}{\partial t}(t\tilde{u}) + D_1\mathcal{L}_\varepsilon(t\tilde{u}) &= \tilde{u} - \frac{D_1}{g^2}\nabla\cdot\Big(\alpha_1(t\tilde{u})g^2JJ^\tau\nabla(\phi - L_3^\varepsilon) + \alpha_1(u - L_1^\varepsilon)g^2JJ^\tau\nabla(t\tilde{\phi})\Big),\\
\frac{\partial}{\partial t}(t\tilde{v}) + D_2\mathcal{L}_\varepsilon(t\tilde{v}) &= \tilde{v} + \frac{D_2}{g^2}\nabla\cdot\Big(\alpha_2(t\tilde{v})g^2JJ^\tau\nabla(\phi - L_3^\varepsilon) + \alpha_2(v - L_2^\varepsilon)g^2JJ^\tau\nabla(t\tilde{\phi})\Big).
\end{aligned} \tag{3.30}$$

The first equation in (3.30) gives

$$\|\mathcal{L}_\varepsilon(t\tilde{\phi})\|_{H_\varepsilon} \le C\left(\|t\tilde{u}\|_{H_\varepsilon} + \|t\tilde{v}\|_{H_\varepsilon}\right). \tag{3.31}$$

Taking the inner product of the second equation in (3.30) with $t\tilde{u}$ in H_ε, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|t\tilde{u}\|_{H_\varepsilon}^2 + D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 =& D_1\alpha_1\int\frac{g^2}{\varepsilon^2}t\tilde{u}J^\tau\nabla(\phi - L_3^\varepsilon)\cdot J^\tau\nabla(t\tilde{u})\\
&+ D_1\alpha_1\int\frac{g^2}{\varepsilon^2}(u - L_1^\varepsilon)J^\tau\nabla(t\tilde{\phi})\cdot J^\tau\nabla(t\tilde{u}) + t\|\tilde{u}\|_{H_\varepsilon}^2.
\end{aligned} \tag{3.32}$$

By (3.29), the first term on the right-hand side of (3.32) is bounded by

$$\begin{aligned}
C\|t\tilde{u}\|_3\|J^\tau\nabla(\phi - L_3^\varepsilon)\|_6\|J^\tau\nabla(t\tilde{u})\|_2 &\le C\|t\tilde{u}\|_2^{\frac{1}{2}}\|t\tilde{u}\|_{H^1}^{\frac{1}{2}}\|J^\tau\nabla(\phi - L_3^\varepsilon)\|_{H^1}\|J^\tau\nabla(t\tilde{u})\|_2\\
&\le C\|t\tilde{u}\|_{H_\varepsilon}^{\frac{1}{2}}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^{\frac{3}{2}}\left(\|\mathcal{L}_\varepsilon\phi\|_{H_\varepsilon} + \|\mathcal{L}_\varepsilon L_3^\varepsilon\|_{H_\varepsilon}\right)\\
&\le C\|t\tilde{u}\|_{H_\varepsilon}^{\frac{1}{2}}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^{\frac{3}{2}} \le \frac{1}{8}D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 + C\|t\tilde{u}\|_{H_\varepsilon}^2.
\end{aligned} \tag{3.33}$$

By (3.31), the second term on the right-hand side of (3.32) is less than

$$\begin{aligned}
C\|u - L_1^\varepsilon\|_\infty\|J^\tau\nabla(t\tilde{\phi})\|_2\|J^\tau\nabla(t\tilde{u})\|_2 &\le \frac{1}{8}D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 + C\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{\phi})\|_{H_\varepsilon}^2\\
&\le \frac{1}{8}D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 + C\left(\|t\tilde{u}\|_{H_\varepsilon}^2 + \|t\tilde{v}\|_{H_\varepsilon}^2\right).
\end{aligned} \tag{3.34}$$

Multiplying the second equation in (3.13) by $t\tilde{u}$, after simple computations, we find that the last term on the right-hand side of (3.32) satisfies

$$\begin{aligned}
t\|\tilde{u}\|_{H_\varepsilon}^2 \le& C\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}}u\|_{H_\varepsilon} + \|\mathcal{L}_\varepsilon^{\frac{1}{2}}L_1^\varepsilon\|_{H_\varepsilon} + \|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_{H_\varepsilon} + \|\mathcal{L}_\varepsilon^{\frac{1}{2}}L_3^\varepsilon\|_{H_\varepsilon}\right)\\
\le& \frac{1}{8}D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 + C.
\end{aligned} \tag{3.35}$$

Combining the estimates in (3.32)-(3.35), we get

$$\frac{d}{dt}\|t\tilde{u}\|_{H_\varepsilon}^2 + D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 \le C\left(\|t\tilde{u}\|_{H_\varepsilon}^2 + \|t\tilde{v}\|_{H_\varepsilon}^2\right) + C. \tag{3.36}$$

Similarly,

$$\frac{d}{dt}\|t\tilde{v}\|_{H_\varepsilon}^2 + D_1\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{v})\|_{H_\varepsilon}^2 \le C\left(\|t\tilde{u}\|_{H_\varepsilon}^2 + \|t\tilde{v}\|_{H_\varepsilon}^2\right) + C. \tag{3.37}$$

Finally, from (3.36)-(3.37), we have

$$\frac{d}{dt}\left(\|t\tilde{u}\|_{H_\varepsilon}^2 + \|t\tilde{v}\|_{H_\varepsilon}^2\right) + C_1\left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{u})\|_{H_\varepsilon}^2 + \|\mathcal{L}_\varepsilon^{\frac{1}{2}}(t\tilde{v})\|_{H_\varepsilon}^2\right) \le C\left(\|t\tilde{u}\|_{H_\varepsilon}^2 + \|t\tilde{v}\|_{H_\varepsilon}^2\right) + C,$$

which, along with Gronwall's lemma, concludes the proof. □

We now describe the analogue of Lemma 3.6 for the one-dimensional limiting system (3.6)-(3.8).

Lemma 3.7. *Given $R > 0$, there exists K depending only on R such that for any $(u_0, v_0) \in \Sigma_0$ with $\|(u_0, v_0)\|_{H^1\times H^1} \le R$, the solution (u, v) of problem (3.6)-(3.8) satisfies*

$$t^2\left(\|\frac{\partial\phi}{\partial t}\|_{H^2}^2 + \|\frac{\partial u}{\partial t}\|_2^2 + \|\frac{\partial v}{\partial t}\|_2^2\right) + \int_0^t s^2\left(\|\frac{\partial u}{\partial s}\|_{H^1}^2 + \|\frac{\partial v}{\partial s}\|_{H^1}^2\right) ds \le Ke^{Kt}, \quad t \ge 0.$$

Proof. The proof is similar to that of Lemma 3.6 but simpler, and therefore omitted here. □

3.3 Upper Semicontinuity

In this section, we establish the upper semicontinuity of global attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$. We first compare the solutions of the three-dimensional problem (3.2)-(3.4) and the one-dimensional limiting problem (3.6)-(3.8), and then establish the relationships between the global attractors of the two dynamical systems.

In what follows, we reformulate limiting system (3.6) as an operator equation. Let H_0 be the $L^2(0,1)$ space with the inner product $(\cdot,\cdot)_{H_0}$ given by

$$(u, v)_{H_0} = \int_0^1 g_0^2 uv\, dx,$$

and let $a_0(\cdot,\cdot)$ be the bilinear form on $\left(H_0^1(0,1)\right)^2$:

$$a_0(w_1, w_2) = \left(\frac{dw_1}{dx}, \frac{dw_2}{dx}\right)_{H_0} = \int_0^1 g_0^2 \frac{dw_1}{dx}\frac{dw_2}{dx}\, dx.$$

For $f \in L^2(\Omega)$, let $M(f) \in L^2(0,1)$ be the function:

$$(M(f))(x) = \frac{1}{\pi}\int_{\mathbb{D}} f(x,y,z)\, dy\, dz.$$

Lemma 3.8. *Suppose $f \in H^1(\Omega)$. Then we have*

$$\|f - M(f)\|_{H_\varepsilon} \le C\varepsilon \|f\|_{X_\varepsilon}.$$

Proof. Notice that

$$\|f - M(f)\|_2^2 = \int_{x=0}^1 \int_{\mathbb{D}} \left| f(x,y,z) - \frac{1}{\pi}\int_{\mathbb{D}} f(x,u,v)dudv \right|^2 dydzdx. \tag{3.38}$$

Using the identity

$$\begin{aligned} f(x, r\cos\theta, r\sin\theta) =& f(x,\rho\cos\phi,\rho\sin\phi) - \int_\theta^\phi \frac{\partial}{\partial t} f(x,\rho\cos t,\rho\sin t)dt \\ & - \int_r^\rho \frac{\partial}{\partial\tau} f(x,\tau\cos\theta,\tau\sin\theta)d\tau, \end{aligned}$$

one can write

$$\begin{aligned} &\frac{1}{\pi}\int_{r=0}^1\int_{\theta=0}^{2\pi} f(x,r\cos\theta,r\sin\theta)rdrd\theta = f(x,\rho\cos\phi,\rho\sin\phi) \\ &\quad - \frac{1}{\pi}\int_{r=0}^1\int_{\theta=0}^{2\pi}\left(\int_{t=\theta}^{\phi}\left(-\rho\sin t\frac{\partial f}{\partial y}(x,\rho\cos t,\rho\sin t) + \rho\cos t\frac{\partial f}{\partial z}(x,\rho\cos t,\rho\sin t)\right)dt\right)rdrd\theta \\ &\quad - \frac{1}{\pi}\int_{r=0}^1\int_{\theta=0}^{2\pi}\left(\int_{\tau=r}^{\rho}\left(\cos\theta\frac{\partial f}{\partial y}(x,\tau\cos\theta,\tau\sin\theta) + \sin\theta\frac{\partial f}{\partial z}(x,\tau\cos\theta,\tau\sin\theta)\right)d\tau\right)rdrd\theta \end{aligned}$$

Then, after simple computations, Lemma 3.8 follows from (3.38) and the above. □

Let $(\psi, P, Q) \in (H_D^1(\Omega))^3$ and let $(\phi_\varepsilon, u_\varepsilon, v_\varepsilon)$ be a solution of system (3.2). In view of the boundary condition (3.3) and the choices of L_k^ε for $k = 1,2,3$, we have

$$\begin{aligned} -a_\varepsilon(\phi_\varepsilon - L_3^\varepsilon, \psi) =& -\lambda\alpha_1(u_\varepsilon - L_1^\varepsilon,\psi)_{H_\varepsilon} + \lambda\alpha_2(v_\varepsilon - L_2^\varepsilon,\psi)_{H_\varepsilon}, \\ \frac{1}{D_1}\left(\frac{\partial u_\varepsilon}{\partial t}, P\right)_{H_\varepsilon} =& -a_\varepsilon(u_\varepsilon - L_1^\varepsilon, P) + \alpha_1\left((u_\varepsilon - L_1^\varepsilon)\mathcal{L}_\varepsilon^{1/2}(\phi_\varepsilon - L_3^\varepsilon), \mathcal{L}_\varepsilon^{1/2}P\right)_{H_\varepsilon}, \\ \frac{1}{D_2}\left(\frac{\partial v_\varepsilon}{\partial t}, Q\right)_{H_\varepsilon} =& -a_\varepsilon(v_\varepsilon - L_2^\varepsilon, Q) - \alpha_2\left((v_\varepsilon - L_2^\varepsilon)\mathcal{L}_\varepsilon^{1/2}(\phi_\varepsilon - L_3^\varepsilon), \mathcal{L}_\varepsilon^{1/2}Q\right)_{H_\varepsilon}. \end{aligned} \tag{3.39}$$

Let (ϕ, u, v) be the solution of the limiting system (3.6). View (ϕ, u, v) as an element in $(H_D^1(\Omega))^3$. Then a direct computation yields that, for $(\psi, P, Q) \in (H_D^1(\Omega))^3$,

$$\begin{aligned} -a_\varepsilon(\phi - L_3^0, \psi) =& -\lambda\alpha_1(u - L_1^0,\psi)_{H_\varepsilon} + \lambda\alpha_2(v - L_2^0,\psi)_{H_\varepsilon} + F(\phi - L_3^0, \psi), \\ \frac{1}{D_1}\left(\frac{\partial u}{\partial t}, P\right)_{H_\varepsilon} =& -a_\varepsilon(u - L_1^0, P) + \alpha_1\left((u - L_1^0)\mathcal{L}_\varepsilon^{1/2}(\phi - L_3^0), \mathcal{L}_\varepsilon^{1/2}P\right)_{H_\varepsilon} \\ & + G_1(u - L_1^0, \phi - L_3^0, P), \\ \frac{1}{D_2}\left(\frac{\partial v}{\partial t}, Q\right)_{H_\varepsilon} =& -a_\varepsilon(v - L_2^0, Q) - \alpha_2\left((v - L_2^0)\mathcal{L}_\varepsilon^{1/2}(\phi - L_3^0), \mathcal{L}_\varepsilon^{1/2}Q\right)_{H_\varepsilon} \\ & + G_2(v - L_2^0, \phi - L_3^0, Q), \end{aligned} \tag{3.40}$$

where, for appropriate functions p, q and r, and for $i = 1, 2$,

$$\begin{aligned} F(p,q) =& \left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) p_x, q \right)_{H_\varepsilon} + \left(\frac{g_x}{g} p_x, y q_y + z q_z \right)_{H_\varepsilon}, \\ G_i(p,q,r) =& - \left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) p_x, r \right)_{H_\varepsilon} - \left(\frac{g_x}{g} p_x, y r_y + z r_z \right)_{H_\varepsilon} \\ &+ (-1)^{i+1} \alpha_i \left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) p q_x, r \right)_{H_\varepsilon} + (-1)^{i+1} \alpha_i \left(\frac{g_x}{g} p q_x, y r_y + z r_z \right)_{H_\varepsilon}. \end{aligned} \tag{3.41}$$

Let

$$\psi^\varepsilon = \phi_\varepsilon - L_3^\varepsilon - (\phi - L_3^0),\ P^\varepsilon = u_\varepsilon - L_1^\varepsilon - (u - L_1^0),\ Q^\varepsilon = v_\varepsilon - L_2^\varepsilon - (v - L_2^0). \tag{3.42}$$

Upon subtracting (3.40) from (3.39), we obtain that for any $(\psi, P, Q) \in H_D^1(\Omega)^3$,

$$a_\varepsilon(\psi^\varepsilon, \psi) = \lambda \alpha_1 (P^\varepsilon, \psi)_{H_\varepsilon} - \lambda \alpha_2 (Q^\varepsilon, \psi)_{H_\varepsilon} + F(\phi - L_3^0, \psi), \tag{3.43}$$

$$\begin{aligned} \frac{1}{D_1} (\partial_t P^\varepsilon, P)_{H_\varepsilon} =& - a_\varepsilon(P^\varepsilon, P) + \alpha_1 \left((u_\varepsilon - L_1^\varepsilon) \mathcal{L}_\varepsilon^{\frac{1}{2}} \psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} P \right)_{H_\varepsilon} \\ &+ \alpha_1 \left(P^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} P \right)_{H_\varepsilon} - G_1(u - L_1^0, \phi - L_3^0, P), \end{aligned} \tag{3.44}$$

$$\begin{aligned} \frac{1}{D_2} (\partial_t Q^\varepsilon, Q)_{H_\varepsilon} =& - a_\varepsilon(Q^\varepsilon, Q) - \alpha_2 \left((v_\varepsilon - L_2^\varepsilon) \mathcal{L}_\varepsilon^{\frac{1}{2}} \psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} Q \right)_{H_\varepsilon} \\ &- \alpha_2 \left(Q^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} Q \right)_{H_\varepsilon} - G_2(v - L_2^0, \phi - L_3^0, Q). \end{aligned} \tag{3.45}$$

For the above system, we have the following estimates.

Lemma 3.9. *There exists $\varepsilon_1 > 0$ such that, for any $R > 0$, there exists a constant K depending on R such that, for any $0 < \varepsilon \le \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$ with $\|(u_0, v_0)\|_{X_\varepsilon \times X_\varepsilon} \le R$, the following holds:*

$$\|P^\varepsilon(t)\|_{H_\varepsilon}^2 + \|Q^\varepsilon(t)\|_{H_\varepsilon}^2 + \|\psi^\varepsilon(t)\|_{X_\varepsilon}^2 + \int_0^t \left(\|P^\varepsilon(s)\|_{X_\varepsilon}^2 + \|Q^\varepsilon(s)\|_{X_\varepsilon}^2 \right) ds \le \varepsilon K e^{Kt}, \quad t \ge 0,$$

where $(\phi_\varepsilon, u_\varepsilon, v_\varepsilon)$ is the solution of problem (3.2)-(3.4) with the initial condition (u_0, v_0), (ϕ, u, v) is the solution of problem (3.6)-(3.8) with the initial condition $(M(u_0), M(v_0))$, and $(\psi^\varepsilon, P^\varepsilon, Q^\varepsilon)$ is given by (3.42).

Proof. It follows from (3.43)–(3.45) that

$$
\begin{aligned}
a_\varepsilon(\psi^\varepsilon,\psi^\varepsilon) =& \lambda\alpha_1(P^\varepsilon,\psi^\varepsilon)_{H_\varepsilon} - \lambda\alpha_2(Q^\varepsilon,\psi^\varepsilon)_{H_\varepsilon} + F(\phi - L_3^0,\psi^\varepsilon), && (3.46)\\
\frac{1}{D_1}\left(\partial_t P^\varepsilon, P^\varepsilon\right)_{H_\varepsilon} =& - a_\varepsilon(P^\varepsilon,P^\varepsilon) + \alpha_1\left((u_\varepsilon - L_1^\varepsilon)\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\right)_{H_\varepsilon} \\
& + \alpha_1\left(P^\varepsilon\mathcal{L}_\varepsilon^{\frac{1}{2}}(\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\right)_{H_\varepsilon} - G_1(u - L_1^0, \phi - L_3^0, P^\varepsilon), && (3.47)\\
\frac{1}{D_2}\left(\partial_t Q^\varepsilon, Q^\varepsilon\right)_{H_\varepsilon} =& - a_\varepsilon(Q^\varepsilon,Q^\varepsilon) - \alpha_2\left((v_\varepsilon - L_2^\varepsilon)\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\right)_{H_\varepsilon} \\
& - \alpha_2\left(Q^\varepsilon\mathcal{L}_\varepsilon^{\frac{1}{2}}(\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\right)_{H_\varepsilon} - G_2(v - L_2^0, \phi - L_3^0, Q^\varepsilon). && (3.48)
\end{aligned}
$$

Next, we estimate each term on the right-hand sides of (3.46)-(3.48). The first two terms on the right-hand side of (3.46) are bounded by:

$$
\begin{aligned}
\lambda\alpha_1\left|(P^\varepsilon,\psi^\varepsilon)_{H_\varepsilon}\right| + \lambda\alpha_2\left|(Q^\varepsilon,\psi^\varepsilon)_{H_\varepsilon}\right| \leq& C\left(\|P^\varepsilon\|_{H_\varepsilon} + \|Q^\varepsilon\|_{H_\varepsilon}\right)\|\psi^\varepsilon\|_{H_\varepsilon} \\
\leq& C\left(\|P^\varepsilon\|^2_{H_\varepsilon} + \|Q^\varepsilon\|^2_{H_\varepsilon}\right) + \frac{1}{4}a_\varepsilon(\psi^\varepsilon,\psi^\varepsilon). \qquad (3.49)
\end{aligned}
$$

By (3.10) we find that g satisfies

$$
\left|\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2}\right| \leq C\varepsilon.
$$

Then, by Lemma 3.5, the first term in $F(\phi - L_3^0,\psi^\varepsilon)$ on the right-hand side of (3.46) is less than

$$
\left|\left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2}\right)(\phi - L_3^0)_x, \psi^\varepsilon\right)_{H_\varepsilon}\right| \leq C\varepsilon\|\psi^\varepsilon\|^2_{H_\varepsilon} \leq C\varepsilon^2 + \frac{1}{4}a_\varepsilon(\psi^\varepsilon,\psi^\varepsilon). \qquad (3.50)
$$

It follows from (3.29) and Lemma 3.5 that the second term in $F(\phi - L_3^0,\psi^\varepsilon)$ on the right-hand side of (3.46) is bounded by, for $t \geq 0$,

$$
\begin{aligned}
&\left|\left(\frac{g_x}{g}(\phi - L_3^0)_x, y\partial_y(\phi_\varepsilon - L_3^\varepsilon) + z\partial_z(\phi_\varepsilon - L_3^\varepsilon)\right)_{H_\varepsilon}\right| \\
&\leq C\left(\|\partial_y\phi_\varepsilon\|_{H_\varepsilon} + \|\partial_z\phi_\varepsilon\|_{H_\varepsilon} + \|\partial_y L_3^\varepsilon\|_{H_\varepsilon} + \|\partial_z L_3^\varepsilon\|_{H_\varepsilon}\right) \leq C\varepsilon. \qquad (3.51)
\end{aligned}
$$

By (3.46) and (3.49)-(3.51), we obtain, for all $t \geq 0$,

$$
\|\psi^\varepsilon(t)\|^2_{X_\varepsilon} \leq Ca_\varepsilon(\psi^\varepsilon,\psi^\varepsilon) \leq C\left(\|P^\varepsilon\|^2_{H_\varepsilon} + \|Q^\varepsilon\|^2_{H_\varepsilon}\right) + C\varepsilon. \qquad (3.52)
$$

We now deal with the right-hand side of (3.47). By (3.52), the second term on the right-hand side of (3.47) is less than

$$
\alpha_1\left|\left((u_\varepsilon - L_1)\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\right)_{H_\varepsilon}\right| \leq C(\|\psi^\varepsilon\|^2_{X_\varepsilon} + \|P^\varepsilon\|^2_{X_\varepsilon}) \leq C(\|P^\varepsilon\|^2_{X_\varepsilon} + \|Q^\varepsilon\|^2_{X_\varepsilon}) + C\varepsilon. \qquad (3.53)
$$

Since the functions ϕ and L_3^0 depend on $x \in (0,1)$ only, we have

$$\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi = J^\tau \nabla\phi = (\partial_x\phi, 0, 0)^\tau, \qquad \mathcal{L}_\varepsilon^{\frac{1}{2}} L_3^0 = J^\tau \nabla L_3^0 = (\partial_x L_3^0, 0, 0)^\tau,$$

which, along with Lemma 3.5 and the first equation of (3.6), implies that, for all $t \geq 0$,

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_\infty = \|\partial_x\phi\|_\infty \leq C\|\partial_x\phi\|_{H^1} \leq C\|\phi\|_{H^2} \leq C. \tag{3.54}$$

By (3.54), the third term on the right-hand side of (3.47) is bounded by

$$\begin{aligned} \alpha_1 \left| \left(P^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}}(\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} P^\varepsilon \right)_{H_\varepsilon} \right| &\leq \alpha_1 \left(\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi\|_\infty + \|\mathcal{L}_\varepsilon^{\frac{1}{2}} L_3^0\|_\infty \right) \|P^\varepsilon\|_{H_\varepsilon} \|\mathcal{L}_\varepsilon^{\frac{1}{2}} P^\varepsilon\|_{H_\varepsilon} \\ &\leq C\|P^\varepsilon\|_{H_\varepsilon}^2 + \frac{1}{4} a_\varepsilon(P^\varepsilon, P^\varepsilon). \end{aligned} \tag{3.55}$$

Note that the term G_1 on the right-hand side of (3.47) can be estimated in a similar manner as (3.49)-(3.51). Therefore, it follows from (3.47) and (3.52)-(3.55) that, for $t \geq 0$,

$$\frac{d}{dt}\|P^\varepsilon\|_{H_\varepsilon}^2 + \|P^\varepsilon\|_{X_\varepsilon}^2 \leq C(\|P^\varepsilon\|_{H_\varepsilon}^2 + \|Q^\varepsilon\|_{H_\varepsilon}^2) + C\varepsilon. \tag{3.56}$$

Similarly, Q^ε satisfies, for $t \geq 0$,

$$\frac{d}{dt}\|Q^\varepsilon\|_{H_\varepsilon}^2 + \|Q^\varepsilon\|_{X_\varepsilon}^2 \leq C(\|P^\varepsilon\|_{H_\varepsilon}^2 + \|Q^\varepsilon\|_{H_\varepsilon}^2) + C\varepsilon. \tag{3.57}$$

Then, it follows from (3.56)-(3.57) that, for $t \geq 0$,

$$\frac{d}{dt}\left(\|P^\varepsilon\|_{H_\varepsilon}^2 + \|Q^\varepsilon\|_{H_\varepsilon}^2\right) + \|P^\varepsilon\|_{X_\varepsilon}^2 + \|Q^\varepsilon\|_{X_\varepsilon}^2 \leq C(\|P^\varepsilon\|_{H_\varepsilon}^2 + \|Q^\varepsilon\|_{H_\varepsilon}^2) + C\varepsilon. \tag{3.58}$$

By Gronwall's lemma, we get

$$\begin{aligned} &\|P^\varepsilon(t)\|_{H_\varepsilon}^2 + \|Q^\varepsilon(t)\|_{H_\varepsilon}^2 \leq e^{Ct}\left(\|P^\varepsilon(0)\|_{H_\varepsilon}^2 + \|Q^\varepsilon(0)\|_{H_\varepsilon}^2\right) + \varepsilon e^{Ct} \\ &\leq Ce^{Ct}\left(\|u_0 - M(u_0)\|_{H_\varepsilon}^2 + \|L_1^\varepsilon - L_1^0\|_{H_\varepsilon}^2 + \|v_0 - M(v_0)\|_{H_\varepsilon}^2 + \|L_2^\varepsilon - L_2^0\|_{H_\varepsilon}^2\right) + \varepsilon e^{Ct}. \end{aligned} \tag{3.59}$$

By (3.1) we see that $L_1, L_2 \in W^{1,\infty}(\Omega_\varepsilon)$, and hence, for $k = 1, 2$,

$$\|L_k^\varepsilon - L_k^0\|_{H_\varepsilon}^2 = \| \int_0^1 \left(yg\frac{\partial L_k}{\partial Y}(x, sgy, sgz) + zg\frac{\partial L_k}{\partial Z}(x, sgy, sgz)\right) ds\|_{H_\varepsilon}^2 \leq C\varepsilon^2. \tag{3.60}$$

From (3.59)-(3.60) and Lemma 3.8, we find that

$$\|P^\varepsilon(t)\|_{H_\varepsilon}^2 + \|Q^\varepsilon(t)\|_{H_\varepsilon}^2 \leq \varepsilon(C+1)e^{Ct}. \tag{3.61}$$

Integrating (3.58) between 0 and t, by (3.61) we conclude Lemma 3.9. $\square$

Next, we improve the uniform estimates in ε given in Lemma 3.9.

Lemma 3.10. *There exists $\varepsilon_1 > 0$ such that, for any $R > 0$, there exists a constant K depending on R such that, for any $0 < \varepsilon \le \varepsilon_1$ and $(u_0, v_0) \in \Sigma_\varepsilon$ with $\|(u_0, v_0)\|_{X_\varepsilon \times X_\varepsilon} \le R$, the following holds:*

$$t^2 \left(\|\frac{\partial P^\varepsilon}{\partial t}\|_{H_\varepsilon}^2 + \|\frac{\partial Q^\varepsilon}{\partial t}\|_{H_\varepsilon}^2 \right) + t \left(\|P^\varepsilon\|_{X_\varepsilon}^2 + \|Q^\varepsilon\|_{X_\varepsilon}^2 \right) \le \sqrt{\varepsilon} K e^{Kt}, \quad t \ge 0,$$

where $(\psi^\varepsilon, P^\varepsilon, Q^\varepsilon)$ is given by (3.42), $(\phi_\varepsilon, u_\varepsilon, v_\varepsilon)$ is the solution of problem (3.2)-(3.4) with the initial condition (u_0, v_0), and (ϕ, u, v) is the solution of problem (3.6)-(3.8) with the initial condition $(M(u_0), M(v_0))$.

Proof. Denote by

$$\tilde{P}^\varepsilon = \frac{\partial P^\varepsilon}{\partial t}, \quad \tilde{Q}^\varepsilon = \frac{\partial Q^\varepsilon}{\partial t}, \quad \tilde{\psi}^\varepsilon = \frac{\partial \psi^\varepsilon}{\partial t}. \tag{3.62}$$

Differentiating systems (3.43)–(3.45) with respect to t, multiplying the resulting systems by t, replacing ψ, P and Q by $t\tilde{\psi}$, $t\tilde{P}$ and $t\tilde{Q}$, respectively, we obtain

$$a_\varepsilon(t\tilde{\psi}^\varepsilon, t\tilde{\psi}^\varepsilon) = \lambda\alpha_1(t\tilde{P}^\varepsilon, t\tilde{\psi}^\varepsilon)_{H_\varepsilon} - \lambda\alpha_2(t\tilde{Q}^\varepsilon, t\tilde{\psi}^\varepsilon)_{H_\varepsilon} + tF(\phi_t, t\tilde{\psi}^\varepsilon), \tag{3.63}$$

$$\begin{aligned}
\frac{1}{2D_1}\frac{d}{dt}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + a_\varepsilon(t\tilde{P}^\varepsilon, t\tilde{P}^\varepsilon) = {}& \alpha_1 t \left(\partial_t u_\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} \psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon \right)_{H_\varepsilon} + \alpha_1 t \left((u_\varepsilon - L_1^\varepsilon) \mathcal{L}_\varepsilon^{\frac{1}{2}} \tilde{\psi}^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon \right)_{H_\varepsilon} \\
& + \alpha_1 t \left(\tilde{P}^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon \right)_{H_\varepsilon} + \alpha_1 t \left(P^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} \phi_t, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon \right)_{H_\varepsilon} \\
& - \alpha_1 t \left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) (u - L_1^0)\phi_{tx}, t\tilde{P}^\varepsilon \right)_{H_\varepsilon} \\
& - \alpha_1 t \left(\frac{g_x}{g} (u - L_1^0)\phi_{tx}, ty\partial_y \tilde{P}^\varepsilon + tz\partial_z \tilde{P}^\varepsilon \right)_{H_\varepsilon} \\
& - tG_1(u_t, \phi - L_3^0, t\tilde{P}^\varepsilon) + \frac{1}{D_1}(\tilde{P}^\varepsilon, t\tilde{P}^\varepsilon)_{H_\varepsilon},
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\frac{1}{2D_2}\frac{d}{dt}\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2 + a_\varepsilon(t\tilde{Q}^\varepsilon, t\tilde{Q}^\varepsilon) = {}& -\alpha_2 t \left(\partial_t v_\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} \psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{Q}^\varepsilon \right)_{H_\varepsilon} - \alpha_2 t \left((v_\varepsilon - L_2^\varepsilon) \mathcal{L}_\varepsilon^{\frac{1}{2}} \tilde{\psi}^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{Q}^\varepsilon \right)_{H_\varepsilon} \\
& - \alpha_2 t \left(\tilde{Q}^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} (\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{Q}^\varepsilon \right)_{H_\varepsilon} - \alpha_2 t \left(Q^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}} \phi_t, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{Q}^\varepsilon \right)_{H_\varepsilon} \\
& + \alpha_2 t \left(\left(\frac{\partial_x g^2}{g^2} - \frac{\partial_x g_0^2}{g_0^2} \right) (v - L_2^0)\phi_{tx}, t\tilde{Q}^\varepsilon \right)_{H_\varepsilon} \\
& + \alpha_2 t \left(\frac{g_x}{g} (v - L_2^0)\phi_{tx}, ty\partial_y \tilde{Q}^\varepsilon + tz\partial_z \tilde{Q}^\varepsilon \right)_{H_\varepsilon} \\
& - tG_2(v_t, \phi - L_3^0, t\tilde{Q}^\varepsilon) + \frac{1}{D_2}(\tilde{Q}^\varepsilon, t\tilde{Q}^\varepsilon)_{H_\varepsilon}.
\end{aligned} \tag{3.65}$$

We now estimate every term involved in the above system. Note that (3.63) implies that

$$\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{\psi}^\varepsilon\|_{H_\varepsilon}^2 \le C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right) + C\varepsilon^2 t^2. \tag{3.66}$$

By (3.29) and Lemma 3.9, we see that the first term on the right-hand side of (3.64) is bounded by

$$\begin{aligned}
&|\alpha_1 t\left(\partial_t u_\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\right)_{H_\varepsilon}| \le Ct\|\partial_t u_\varepsilon\|_6 \|\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon\|_3 \|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_2 \\
&\le Ct\|\partial_t u_\varepsilon\|_{H^1} \|\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon\|_2^{\frac{1}{2}} \|\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon\|_{H^1}^{\frac{1}{2}} \|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_2 \\
&\le \frac{1}{32}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + Ct^2\|\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_t u_\varepsilon\|_{H_\varepsilon}^2 \|\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon\|_{H_\varepsilon} \|\mathcal{L}_\varepsilon\psi^\varepsilon\|_{H_\varepsilon} \\
&\le \frac{1}{32}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \sqrt{\varepsilon} Ce^{Ct}\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_t u_\varepsilon\|_{H_\varepsilon}^2.
\end{aligned} \tag{3.67}$$

By (3.66), the second term on the right-hand side of (3.64) is less than

$$\alpha_1 t\left((u_\varepsilon - L_1^\varepsilon)\mathcal{L}_\varepsilon^{\frac{1}{2}}\tilde{\psi}^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\right)_{H_\varepsilon} \le \frac{1}{32}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right) + \varepsilon^2 Ct^2. \tag{3.68}$$

By Lemma 3.7, the fourth term on the right-hand side of (3.64) is bounded by

$$C\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\phi_t\|_\infty \|P^\varepsilon\|_2 \|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_2 \le \frac{1}{32}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + C\|t\phi_t\|_{H^2}^2 \|P^\varepsilon\|_{H_\varepsilon}^2 \le \frac{1}{32}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \varepsilon Ce^{Ct}. \tag{3.69}$$

Other terms on the right-hand side of (3.64) can be estimated in a similar way as the proof of Lemma 3.9. Therefore, by (3.64), (3.67)-(3.69) and the estimates for other terms, we have

$$\begin{aligned}
\frac{1}{2D_1}\frac{d}{dt}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \frac{3}{4}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 \le& C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right) + \varepsilon Ce^{Ct} + \varepsilon^2 C\left(\|tu_t\|_{H^1}^2 + \|tv_t\|_{H^1}^2\right) \\
&+ \sqrt{\varepsilon} Ce^{Ct}\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_t u_\varepsilon\|_{H_\varepsilon}^2 + \frac{1}{D_1}(\tilde{P}^\varepsilon, t\tilde{P}^\varepsilon)_{H_\varepsilon}.
\end{aligned} \tag{3.70}$$

Next, we deal with the last term on the right-hand side of the above inequality. Replacing P in (3.44) by $t\partial_t P^\varepsilon = t\tilde{P}^\varepsilon$, we get

$$\begin{aligned}
\frac{1}{D_1}(\tilde{P}^\varepsilon, t\tilde{P}^\varepsilon)_{H_\varepsilon} + \frac{1}{2}t\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} P^\varepsilon\|_{H_\varepsilon}^2 =& \alpha_1\left((u_\varepsilon - L_1^\varepsilon)\mathcal{L}_\varepsilon^{\frac{1}{2}}\psi^\varepsilon, \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\right)_{H_\varepsilon} \\
&+ \alpha_1\left(P^\varepsilon \mathcal{L}_\varepsilon^{\frac{1}{2}}(\phi - L_3^0), \mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\right)_{H_\varepsilon} - G_1(u - L_1^0, \phi - L_3^0, t\tilde{P}^\varepsilon).
\end{aligned}$$

Using Lemma 3.9 and proceeding as before, we obtain from the above that

$$\frac{1}{D_1}(\tilde{P}^\varepsilon, t\tilde{P}^\varepsilon)_{H_\varepsilon} + \frac{1}{2}t\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} P^\varepsilon\|_{H_\varepsilon}^2 \le \frac{1}{4}\|\mathcal{L}_\varepsilon^{\frac{1}{2}} t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2 + \varepsilon Ce^{Ct} + \varepsilon^2 C. \tag{3.71}$$

Then it follows from (3.70)-(3.71) that

$$\begin{aligned}\frac{1}{2D_1}\frac{d}{dt}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+\frac{1}{2}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+\frac{1}{2}t\frac{d}{dt}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2 &\\ \le C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right)+\varepsilon Ce^{Ct}&\\ +\varepsilon^2C\left(\|tu_t\|_{H^1}^2+\|tv_t\|_{H^1}^2\right)+\sqrt{\varepsilon}Ce^{Ct}\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_tu_\varepsilon\|_{H_\varepsilon}^2,&\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{1}{D_1}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2\right)\le \frac{1}{2}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2+C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right)+\varepsilon Ce^{Ct}&\\ +\varepsilon^2C\left(\|tu_t\|_{H^1}^2+\|tv_t\|_{H^1}^2\right)+\sqrt{\varepsilon}Ce^{Ct}\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_tu_\varepsilon\|_{H_\varepsilon}^2.&\end{aligned}\tag{3.72}$$

Similarly, by equation (3.65), we can show that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{1}{D_2}\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\|_{H_\varepsilon}^2\right)\le \frac{1}{2}\|\mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\|_{H_\varepsilon}^2+C\left(\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2\right)+\varepsilon Ce^{Ct}&\\ +\varepsilon^2C\left(\|tu_t\|_{H^1}^2+\|tv_t\|_{H^1}^2\right)+\sqrt{\varepsilon}Ce^{Ct}\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_tv_\varepsilon\|_{H_\varepsilon}^2.&\end{aligned}\tag{3.73}$$

By (3.72)-(3.73) we find that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{D_1}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2+\frac{1}{D_2}\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\|_{H_\varepsilon}^2\right)\\ &\quad\le C\left(\frac{1}{D_1}\|t\tilde{P}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2+\frac{1}{D_2}\|t\tilde{Q}^\varepsilon\|_{H_\varepsilon}^2+t\|\mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\|_{H_\varepsilon}^2\right)+\|\mathcal{L}_\varepsilon^{\frac{1}{2}}P^\varepsilon\|_{H_\varepsilon}^2+\|\mathcal{L}_\varepsilon^{\frac{1}{2}}Q^\varepsilon\|_{H_\varepsilon}^2\\ &\quad+\varepsilon Ce^{Ct}+\varepsilon^2C\left(\|tu_t\|_{H^1}^2+\|tv_t\|_{H^1}^2\right)+\sqrt{\varepsilon}Ce^{Ct}\left(\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_tu_\varepsilon\|_{H_\varepsilon}^2+\|t\mathcal{L}_\varepsilon^{\frac{1}{2}}\partial_tv_\varepsilon\|_{H_\varepsilon}^2\right),\end{aligned}$$

which, along with Gronwall's lemma and Lemmas 3.6, 3.7 and 3.9, implies Lemma 3.10. □

Let $(c_1^\varepsilon, c_2^\varepsilon, \Phi^\varepsilon)$ be the solutions of problem (2.4)-(2.5) with initial datum $(c_{1,0}, c_{2,0})$, and (c_1, c_2, Φ) be the solutions of problem (2.6)-(2.7) with initial datum $(M(c_{1,0}), M(c_{2,0}))$. Then as an immediate consequence of Lemma 3.10, we find the following estimates which are essential to prove the upper semi-continuity of the global attractors.

Lemma 3.11. *There exists $\varepsilon_1 > 0$ such that, for any $R > 0$, there exists a constant K depending on R such that, for any $0 < \varepsilon \le \varepsilon_1$ and $(c_{1,0}, c_{2,0}) \in \tilde{\Sigma}$ with $\|(c_{1,0}, c_{2,0})\|_{X_\varepsilon\times X_\varepsilon} \le R$, the following holds:*

$$\left(\|c_1^\varepsilon(t)-c_1(t)\|_{X_\varepsilon}^2+\|c_2^\varepsilon(t)-c_2(t)\|_{X_\varepsilon}^2\right)\le\sqrt{\varepsilon}Ke^{Kt},\quad t\ge 1.$$

We are now in a position to prove the upper semi-continuity of global attractors.

Proof of Theorem 2.2. Let $T^\varepsilon(t)_{t\geq 0}$ and $T^0(t)_{t\geq 0}$ be the solution operators of problem (2.4)-(2.5) and problem (2.6)-(2.7), respectively. Then it follows from Proposition 3.4 that there is a constant $R > 0$ (independent of ε) such that

$$\|(c_1, c_2)\|_{X_\varepsilon \times X_\varepsilon} \leq R, \quad \text{for all } \ (c_1, c_2) \in \mathcal{A}_\varepsilon.$$

For the given $\eta > 0$, since $\mathcal{A}_0$ is the global attractor of $T^0(t)$, there exists $\tau_0 = \tau_0(\eta, R) \geq 1$ such that, for any $t \geq \tau_0$,

$$\inf_{z_0\in\mathcal{A}_0} \|T^0(t)(Mz) - z_0\|_{X_\varepsilon\times X_\varepsilon} \leq \frac{\eta}{2},$$

for any $z = (c_1, c_2) \in \mathcal{A}_\varepsilon$. On the other hand, by Lemma 3.11 we find that

$$\|T^\varepsilon(\tau_0)z - T^0(\tau_0)(Mz)\|_{X_\varepsilon\times X_\varepsilon} \leq \varepsilon^{\frac{1}{4}} K(R) e^{K(R)\tau_0},$$

for some constant $K(R)$. Therefore, we obtain that, for any $z = (c_1, c_2) \in \mathcal{A}_\varepsilon$:

$$\inf_{z_0\in\mathcal{A}_0} \|T^\varepsilon(\tau_0)z - z_0\|_{X_\varepsilon\times X_\varepsilon} \leq \frac{\eta}{2} + \varepsilon^{\frac{1}{4}} K(R) e^{K(R)\tau_0},$$

which implies that, for $\varepsilon > 0$ small enough:

$$\text{dist}_{X_\varepsilon\times X_\varepsilon} (T^\varepsilon(\tau_0)\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq \eta.$$

The proof is completed since $T^\varepsilon(\tau_0)\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon$.

4 Appendix

The following lemma, which can be verified by direct computations, is used in the derivation of a limiting PNP system in Section 2.1.

Lemma 4.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $\psi(p) = q$, be a diffeomorphism, and let $J(q) = \frac{\partial q}{\partial p}(\psi^{-1}(q))$ be the Jacobian matrix and $d(q) = (\det J(q))^{-1}$. If $\alpha(p) = \beta(\psi(p)) : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then the gradients in the two coordinates are related as*

$$\nabla_p \alpha(p) = J^\tau(q) \nabla_q \beta(q).$$

Further, if $\sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(d(q)\frac{\partial q_j}{\partial p_i}\right) = 0$ for all $i = 1, \cdots, n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, then $F(p) = f(\psi(p))$ satisfies

$$\nabla_p \cdot F(p) = \frac{1}{d(q)} \nabla_q \cdot (d(q)J(q)f(q)),$$

and hence, the Laplace operators are related as

$$\triangle_p \alpha(p) = \frac{1}{d(q)} \nabla_q \cdot (d(q)J(q)J^\tau(q)\nabla_q \beta(q)).$$

The next lemma is used in the homogenization of boundary conditions in Section 3.1.

Lemma 4.2. *Let $h : [0,1] \to \mathbb{R}$ be a smooth function. Then, for any $\varepsilon > 0$, there is a function $H^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ such that $H^\varepsilon(X,0,0) = h(X)$, $H^\varepsilon(0,Y,Z) = h(0)$, $H^\varepsilon(1,Y,Z) = h(1)$, and $\langle \nabla H^\varepsilon(X,Y,Z), \mathbf{n}\rangle = 0$ for $(X,Y,Z) \in \hat{M}_\varepsilon$.*

Proof. We provide a specific construction of a function H^ε. For convenience, hereafter, we denote by $g'(X,\varepsilon) = \frac{\partial g}{\partial X}(X,\varepsilon)$. For any $\varepsilon > 0$ and $X_0 \in [0,1]$, let $X = \psi^\varepsilon(t, X_0)$ be the solution of

$$\frac{dX}{dt} = -t\frac{g'(X,\varepsilon)}{g(X,\varepsilon)} \tag{4.1}$$

with $\psi^\varepsilon(0,X_0) = X_0$. It is easy to see that $\psi^\varepsilon(t,X_0)$ is even in t from the equation. Since $g'(0,\varepsilon) = g'(1,\varepsilon) = 0$, $\psi^\varepsilon(t,0) = 0$ and $\psi^\varepsilon(t,1) = 1$ for all t. Therefore, for any $(X,t) \in [0,1] \times [0, g(X,\varepsilon)]$, there is a unique $X_0 \in [0,1]$ such that $X = \psi^\varepsilon(t,X_0)$, and hence, for any $(X,Y,Z) \in \Omega_\varepsilon$, there is a unique $X_0 \in [0,1]$ such that $X = \psi^\varepsilon(\sqrt{Y^2+Z^2}, X_0)$. Set $H^\varepsilon(X,Y,Z) = h(X_0)$ if $X = \psi^\varepsilon(\sqrt{Y^2+Z^2}, X_0)$. Then, $H^\varepsilon(X,0,0) = h(X)$, $H^\varepsilon(0,Y,Z) = h(0)$ and $H^\varepsilon(1,Y,Z) = h(1)$. It remains to show that, for $(X,Y,Z) \in \hat{M}_\varepsilon$, $\langle \nabla H^\varepsilon(X,Y,Z), \mathbf{n}\rangle = 0$. For any $X_0 \in [0,1]$, the set

$$D(X_0) = \{(X,Y,Z) : X = \psi^\varepsilon(\sqrt{Y^2+Z^2}, X_0)\} = \{(X,Y,Z) : H(X,Y,Z) = h(X_0)\},$$

is a level set of H^ε. Note also that the curve $\{(X,Y,0) : X = \psi^\varepsilon(Y,X_0)\}$ lies on $D(X_0)$ and it is a solution curve to (4.1) if Y is viewed as the t-variable. Therefore, at $(X,Y,0) = (X, g(X,\varepsilon), 0) \in D(X_0) \cap \hat{M}_\varepsilon$, the vector

$$\left(-Y\frac{g'(X,\varepsilon)}{g(X,\varepsilon)}, 1, 0\right) = (-g'(X,\varepsilon), 1, 0)$$

is tangent to $D(X_0)$, and hence, $\langle \nabla H^\varepsilon(X, g(X,\varepsilon), 0), (-g'(X,\varepsilon),1,0)\rangle = 0$. Since $\mathbf{n}$ is parallel to $(-g'(X,\varepsilon),1,0)$, $\langle \nabla H^\varepsilon(X,g(X,\varepsilon),0), \mathbf{n}\rangle = 0$. Due to the rotation symmetry of $\hat{M}_\varepsilon$ and H^ε about the X-axis, we conclude that, for $(X,Y,Z) \in \hat{M}_\varepsilon$, $\langle \nabla H^\varepsilon(X,Y,Z), \mathbf{n}\rangle = 0$. □

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