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QUALITATIVE PROPERTIES OF IONIC FLOWS VIA POISSON-NERNST-PLANCK SYSTEMS WITH BIKERMAN'S LOCAL HARD-SPHERE POTENTIAL: ION SIZE EFFECTS

Yusheng Jia

Department of Mathematics, Qingdao Binhai University, Qingdao, Shandong 266555, China

Weishi Liu

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

Mingji Zhang

Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, NM 87801, USA

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Abstract. We study a quasi-one-dimensional steady-state Poisson-Nernst-Planck model for ionic flows through membrane channels with fixed boundary ion concentrations and electric potentials. We consider two ion species, one positively charged and one negatively charged, and assume zero permanent charge. Bikerman's local hard-sphere potential is included in the model to account for ion size effects on the ionic flow. The model problem is treated as a boundary value problem of a singularly perturbed differential system. Our analysis is based on the geometric singular perturbation theory but, most importantly, on specific structures of this concrete model. The existence of solutions to the boundary value problem for small ion sizes is established and, treating the ion sizes as small parameters, we also derive approximations of individual fluxes and I-V (current-voltage) relations, from which qualitative properties of ionic flows related to ion sizes are studied. A detailed characterization of complicated interactions among multiple and physically crucial parameters for ionic flows, such as boundary concentrations and potentials, diffusion coefficients and ion sizes, is provided.

1. **Introduction.** We study the dynamics of ionic flows, the electrodiffusion of charges, through ion channels via a quasi-one-dimensional steady-state Poisson-Nernst-Planck (PNP) system. As a basic macroscopic model for electrodiffusion of charges, particularly for ionic flows through ion channels ([8, 10, 15, 16, 17, 18, 19, 26, 27, 31, 38, 39, 62, 64, 72, 73, 74], etc.), under various reasonable conditions, PNP systems can be derived as reduced models from molecular dynamic models ([80]), from Boltzmann equations ([2]), and from variational principles ([34, 36, 37]).

The simplest PNP system is the *classical* Poisson-Nernst-Planck (cPNP) system that includes only the ideal components of the electrochemical potentials. It has been simulated ([6, 7, 8, 9, 11, 26, 27, 30, 32, 33, 39, 40, 41, 48, 61, 77, 88]) and analyzed ([1, 3, 4, 20, 21, 24, 43, 53, 55, 56, 59, 65, 75, 76, 81, 82, 83, 84, 86, 87])

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to a great extent. However, a major weakness of the cPNP is that it treats ions as *point-charges*, which is reasonable only for near infinite dilute ionic mixtures. To study the ion size effect on ionic flows, in particular, for ion species with the same valence but different ion sizes, for example, Na^+ (sodium) and K^+ (potassium), one has to consider excess (beyond the ideal) components in the electrochemical potential. One way is to include hard-sphere (HS) potentials. PNP models with ion size effects have been investigated computationally with great successes ([13, 25, 27, 29, 34, 35, 36, 37, 47, 89], etc.), and have been mathematically analyzed (see, for example, [22, 42, 49, 50, 51, 58]).

In [51], the authors provided an analytical treatment of a quasi-one-dimensional version of PNP system with a HS potential. They studied the case where two oppositely charged ions are involved for the same ion channel with *electroneutrality* (zero net charge) boundary conditions, the permanent charge can be ignored, and a *local* HS potential derived from Rosenfeld's *nonlocal* one is included. The authors treated ion sizes as small parameters, and derived an approximation of the I-V relation. Furthermore, the approximate I-V relation allows them to establish the following results.

(i) There is a critical potential V_c so that, if $V > V_c$, then ion sizes *enhance* the current I; if $V < V_c$, then ion sizes *reduce* the current I.
(ii) There is another critical potential V^c so that, if $V > V^c$, then the current I *increases* with respect to $\lambda = d_2/d_1$ where d_1 and d_2 are, respectively, the diameters of the positively and negatively charged ions; if $V < V^c$, then the current I *decreases* in λ.
(iii) Important scaling laws of I-V relations and critical potentials in boundary concentrations are obtained; that is,
    (a) the contribution to the I-V relation from the ideal component scales *linearly* in boundary concentrations;
    (b) the contribution (up to first order in ionic diameters) to the I-V relation from the HS component scales *quadratically* in boundary concentrations;
    (c) both V_c and V^c scale *invariantly* in boundary concentrations.

In this paper, we study a quasi-one-dimensional version of PNP type system with a *local* HS model proposed by Bikerman ([5]). Bikerman's model is one of the *earliest* local models for HS potentials. The problem we study here has basically the same setting as that in [51] except that we take a different *local* model. We study the PNP system with Bikerman's local HS potential for two purposes:

(I) To compare our results with those obtained in [51];
(II) To examine ion size effects on individual fluxes that provide detailed information on the interactions among different ion species within the channel. This is the main contribution compared to [51].

The rest of this paper is organized as follows. In Section 2, we describe the quasi-one-dimensional PNP model of ionic flows, Bikerman's local HS potential, the boundary value problem (BVP) of the singularly perturbed PNP-HS system, and the basic assumptions.

In Section 3, the existence and (local) uniqueness result for the BVP is established in the framework of geometric singular perturbation theory. Based on the analysis in Section 3 and treating the ion sizes as small parameters, approximations of individual fluxes and the I-V (current-voltage) relations are derived, from which the ion size effect on ionic flows is analyzed in detail. This leads to our main interest studied in Section 4, which contains four subsections. In Subsection 4.1, we examine the ion size effect on individual fluxes. We identify *four* critical potentials or voltages, denoted by V_{kc} and V^{kc}, $k = 1, 2$, respectively. The values V_{1c} and V_{2c} are the potentials that balance the ion size effect on the individual fluxes of charge, and the values V^{1c} and V^{2c} are the potentials that separate the relative size effect on the individual fluxes of charge (see Definition 4.4, Theorem 4.7 and Theorem 4.8). More interestingly, under electroneutrality conditions, we observed that $V_{kc} = V^{kc}$ for $k = 1, 2$, while it is not true without electroneutrality conditions (see Corollary 4.5). Subsection 4.2 deals with the ion size effect on the total flow rate of charge (the I-V relations). *Two* critical potentials or voltages V_c and V^c are also identified. In particular, the two critical potential values are identical (see Definition 4.11). The roles of these critical potentials in characterizing ion size effects on ionic flows are discussed. In subsection 4.3, under electroneutrality conditions, the relationship among those critical potentials $V_{kc},\ k = 1, 2$ for individual fluxes and V_c for the I-V relations is studied in terms of multiple physical parameters such as boundary concentrations, boundary potentials and diffusion coefficients (see Lemma 4.19). The distinct effects of the nonlinear interplay between these physical parameters are characterized. In Subsection 4.4, a special case of ion size effects on ionic flows is considered.

We remark that, under electroneutrality boundary conditions, each of these critical potentials separates the potential into two regions over which the ion size effects are qualitatively opposite to each other (see Theorems 4.7, 4.8, and 4.15). Also, in the absence of electroneutrality, it is rather surprising that the roles of critical potentials on ion size effects are significant different: the opposite effects of ion sizes separated by those critical potentials depend on other quantities in terms of boundary conditions (see Proposition 4.18).

Finally, we would like to point out that, under electroneutrality conditions, as what we expect, the results related to ion size effects on the I-V relations are similar to those obtained in [51]. However, our analysis of the ion size effects on the individual fluxes provides detailed information on the interactions among different ion species. We believe our results will provide useful insights for numerical and even experimental studies of ionic flows through membrane channels.

2. **Problem Setup.**

2.1. **A quasi-one-dimensional steady-state PNP type system.**

The channel is assumed to be narrow so that it can be effectively viewed as a one-dimensional channel $[0, l]$ where l, typically in the range of $10 - 20$ nanometers, is the length of the channel together with the baths that the channel links. A quasi-one-dimensional *steady-state* PNP model for ion flows of n ion species though a single channel is (see [57, 61])

$$\begin{aligned} &\frac{1}{A(X)}\frac{d}{dX}\left(\varepsilon_r(X)\varepsilon_0 A(X)\frac{d\Phi}{dX}\right) = -e\left(\sum_{j=1}^{n} z_j C_j(X) + Q(X)\right), \\ &\frac{d\mathcal{J}_i}{dX} = 0, \quad -\mathcal{J}_i = \frac{1}{k_B T}\mathcal{D}_i(X)A(X)C_i(X)\frac{d\mu_i}{dX}, \quad i = 1, 2, \cdots, n \end{aligned} \tag{2.1}$$

where $X \in [0, l]$, e is the elementary charge, k_B is the Boltzmann constant, T is the absolute temperature; Φ is the electric potential, $Q(X)$ is the permanent charge of the channel, $\varepsilon_r(X)$ is the relative dielectric coefficient, ε_0 is the vacuum permittivity;

$A(X)$ is the area of the cross-section of the channel over the point $X \in [0, l]$; for the ith ion species, C_i is the concentration (number of ith ions per volume), z_i is the valence (number of charges per particle) that is positive for cations and negative for anions, μ_i is the electrochemical potential, $\mathcal{J}_i$ is the flux density, and $\mathcal{D}_i(X)$ is the diffusion coefficient. The boundary conditions are, for $i = 1, 2, \cdots, n$,

$$\Phi(0) = \mathcal{V}, \quad C_i(0) = \mathcal{L}_i > 0; \quad \Phi(l) = 0, \quad C_i(l) = \mathcal{R}_i > 0. \tag{2.2}$$

2.2. **Excess potential and a local hard-sphere model.** The electrochemical potential $\mu_i(X)$ for the ith ion species consists of the ideal component $\mu_i^{id}(X)$, the excess component $\mu_i^{ex}(X)$ and the concentration-independent component $\mu_i^0(X)$ (e.g. a hard-well potential):

$$\mu_i(X) = \mu_i^0(X) + \mu_i^{id}(X) + \mu_i^{ex}(X)$$

where

$$\mu_i^{id}(X) = z_i e \Phi(X) + k_B T \ln \frac{C_i(X)}{C_0} \tag{2.3}$$

with some characteristic number density C_0. As mentioned in the introduction, cPNP system takes into consideration of the ideal component $\mu_i^{id}(X)$ only. This component reflects the collision between ion particles and water (medium) molecules. It has been accepted that cPNP system is a reasonable model in, for example, *dilute* cases under which ion particles can be treated as point-charges and ion-to-ion interactions can be more or less ignored. The excess electrochemical potential $\mu_i^{ex}(X)$ accounts for finite size effects of charges (see, e.g., [69, 70]).

In this paper, we study the PNP system including a *local hard-sphere* (LHS) chemical potential, proposed by Bikerman ([5]) to account for the finite size effect. Bikerman's model is, for $i = 1, 2, \ldots, n$,

$$\mu_i^{Bik}(X) = -\, k_B T \ln \left(1 - \sum_{j=1}^{n} v_j C_j(X)\right), \tag{2.4}$$

where v_j is the volume of a *single* jth ion species.

Remark 2.1. Since c_i is the *number density* of ith ion species, it follows that $\sum_{j=1}^{n} v_j c_j < 1$. In this sense, Bikerman's LHS takes into consideration of nonzero ion sizes. It should be pointed out though Bikerman's LHS is not ion specific since it is the same for all ion species.

2.3. **The BVP and assumptions.** The main focus of this paper is to examine the qualitative properties of ion size effects on ionic flows via BVP (2.1)-(2.2) with LHS model (2.4). We will take essentially the same setting as that in [51] except that we use Bikerman's LHS (2.4). More precisely,

(i) We consider two ion species ($n = 2$) with $z_1 > 0$ and $z_2 < 0$;
(ii) The permanent charge is set to be zero: $Q(X) = 0$;
(iii) For the electrochemical potential μ_i, in addition to the ideal component μ_i^{id}, we also include the LHS potential μ_i^{Bik} in (2.4);
(iv) The relative dielectric coefficient and the diffusion coefficients are assumed to be constants, that is, $\varepsilon_r(X) = \varepsilon_r$ and $\mathcal{D}_i(X) = \mathcal{D}_i$.

We first make a dimensionless rescaling following ([24]).

Set $C_0 = \max\{\mathcal{L}_i, \mathcal{R}_i : i = 1, 2\}$ and let

$$\begin{aligned}
\varepsilon^2 =& \frac{\varepsilon_r \varepsilon_0 k_B T}{e^2 l^2 C_0}, \quad x = \frac{X}{l}, \quad h(x) = \frac{A(X)}{l^2}, \quad D_i = l C_0 \mathcal{D}_i; \\
\phi(x) =& \frac{e}{k_B T}\Phi(X), \quad c_i(x) = \frac{C_i(X)}{C_0}, \quad J_i = \frac{\mathcal{J}_i}{D_i}; \\
V =& \frac{e}{k_B T}\mathcal{V}, \quad L_i = \frac{\mathcal{L}_i}{C_0}; \quad R_i = \frac{\mathcal{R}_i}{C_0}.
\end{aligned} \tag{2.5}$$

System (2.1) becomes, with the substitution (2.3) for μ_i^{id},

$$\begin{aligned}
&\frac{\varepsilon^2}{h(x)}\frac{d}{dx}\left(h(x)\frac{d}{dx}\phi\right) = -(z_1 c_1 + z_2 c_2), \quad \frac{dJ_i}{dx} = 0, \\
&h(x)\frac{dc_1}{dx} + h(x) z_1 c_1 \frac{d\phi}{dx} + \frac{h(x)c_1(x)}{k_B T}\frac{d}{dx}\mu_1^{Bik}(x) = -J_1, \\
&h(x)\frac{dc_2}{dx} + h(x) z_2 c_2 \frac{d\phi}{dx} + \frac{h(x)c_2(x)}{k_B T}\frac{d}{dx}\mu_2^{Bik}(x) = -J_2.
\end{aligned} \tag{2.6}$$

It follows from (2.4) that

$$\frac{1}{k_B T}\mu_i^{Bik}(x) = -\ln\Big(1 - \nu_1 c_1(x) - \nu_2 c_2(x)\Big) \ \text{ where } \ \nu_j = v_j C_0. \tag{2.7}$$

Substituting (2.7) into system (2.6), we obtain the BVP

$$\begin{aligned}
&\frac{\varepsilon^2}{h(x)}\frac{d}{dx}\left(h(x)\frac{d}{dx}\phi\right) = -(z_1 c_1 + z_2 c_2), \quad \frac{dJ_i}{dx} = 0, \\
&\frac{dc_1}{dx} = -f_1(c_1, c_2; \nu_1, \nu_2)\frac{d\phi}{dx} - \frac{1}{h(x)} g_1(c_1, J_1, J_2; \nu_1, \nu_2), \\
&\frac{dc_2}{dx} = f_2(c_1, c_2; \nu_1, \nu_2)\frac{d\phi}{dx} - \frac{1}{h(x)} g_2(c_2, J_1, J_2; \nu_1, \nu_2),
\end{aligned} \tag{2.8}$$

with boundary conditions, for $i = 1, 2$,

$$\phi(0) = V, \quad c_i(0) = L_i; \quad \phi(1) = 0, \quad c_i(1) = R_i, \tag{2.9}$$

where

$$\begin{aligned}
f_1(c_1, c_2; \nu_1, \nu_2) =& (z_1 - z_1\nu_1 c_1 - z_2 \nu_2 c_2) c_1, \\
f_2(c_1, c_2; \nu_1, \nu_2) =& -(z_2 - z_1\nu_1 c_1 - z_2 \nu_2 c_2) c_2, \\
g_1(c_1, J_1, J_2; \nu_1, \nu_2) =& J_1 - (\nu_1 J_1 + \nu_2 J_2) c_1, \\
g_2(c_2, J_1, J_2; \nu_1, \nu_2) =& J_2 - (\nu_1 J_1 + \nu_2 J_2) c_2.
\end{aligned} \tag{2.10}$$

For a solution of BVP (2.8)-(2.9), the *total flux of charge or current* $\mathcal{I}$ is

$$\mathcal{I} = z_1 \mathcal{J}_1 + z_2 \mathcal{J}_2 = z_1 D_1 J_1 + z_2 D_2 J_2. \tag{2.11}$$

For fixed L_i's and R_i's, formula (2.11) provides a relation of the current $\mathcal{I}$ on the voltage V. This relation is the so-called *I-V relation* (current-voltage relation).

The BVP (2.8)-(2.9) will be analyzed in Section 3 based on the assumption that the dimensionless parameter ε is small so that system (2.8) can be treated as a singularly perturbed system with ε as the singular parameter. For typical ion channel problems, physical range for the parameter ε is $10^{-2} - 10^{-6}$, which is smaller for crowded ionic mixtures (large C_0) and larger for less crowded ionic mixtures. It is further assumed that the dimensionless parameters ν_i's are small; typical physical range for $\nu_i = v_i C_0$ is $10^{-2} - 10^{-4}$ with 10^{-2} corresponding to crowded ionic mixtures, say, $C_0 \sim 10$ M (molar) and with 10^{-4} to less crowded ionic mixtures, say, $C_0 \sim 100$ mM. From the analysis of the BVP, we will obtain approximations for both J_i's and I-V relations to study ion size effects on ionic flows.

3. Geometry singular perturbation theory for (2.8)–(2.9).

We will rewrite system (2.8) into a standard form for singularly perturbed systems and convert BVP (2.8)-(2.9) to a connecting problem. Generally, there is no unique way to write a second order singular perturbed equation (the Poisson equation for ϕ in (2.8)) to a system of first order equations. Different choices result in different $\epsilon = 0$ limiting systems. It is the $\epsilon = 0$ limiting systems that govern the viable choices. Ideally, one would like to obtain a normally hyperbolic (NH) slow manifold, if possible. The choice used in this paper below (introduced first in [53] to the best of our knowledge) results in a NH slow manifold while the "natural" choice $\dot{\phi} = \varepsilon^2 u$ would not.

Denote the derivative with respect to x by overdot and introduce $u = \varepsilon\dot{\phi}$ and $\tau = x$. System (2.8) becomes

$$\begin{aligned}
\varepsilon\dot{\phi} =& u, \quad \varepsilon\dot{u} = -z_1c_1 - z_2c_2 - \varepsilon\frac{h_\tau(\tau)}{h(\tau)}u,\\
\varepsilon\dot{c}_1 =& -f_1(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_1(c_1, J_1, J_2; \nu_1, \nu_2),\\
\varepsilon\dot{c}_2 =& f_2(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_2(c_2, J_1, J_2; \nu_1, \nu_2)\\
\dot{J}_1 =& \dot{J}_2 = 0, \quad \dot{\tau} = 1.
\end{aligned} \tag{3.1}$$

System (3.1) is the *slow system* and its phase space is $\mathbb{R}^7$ with state variables $(\phi, u, c_1, c_2, J_1, J_2, \tau)$.

For $\varepsilon > 0$, the rescaling $x = \varepsilon\xi$ of the independent variable x gives rise to the *fast system*

$$\begin{aligned}
\phi' =& u, \quad u' = -z_1c_1 - z_2c_2 - \varepsilon\frac{h_\tau(\tau)}{h(\tau)}u,\\
c_1' =& -f_1(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_1(c_1, J_1, J_2; \nu_1, \nu_2),\\
c_2' =& f_2(c_1, c_2; \nu_1, \nu_2)u - \frac{\varepsilon}{h(\tau)}g_2(c_2, J_1, J_2; \nu_1, \nu_2),\\
J_1' =& J_2' = 0, \quad \tau' = \varepsilon,
\end{aligned} \tag{3.2}$$

where prime denotes the derivative with respect to the variable ξ.

For $\varepsilon > 0$, slow system (3.1) and fast system (3.2) have exactly the same phase portrait. But their limiting systems at $\varepsilon = 0$ are different. System of (3.1) with $\varepsilon = 0$ is called the *limiting slow system*, whose orbits are called *slow orbits* or regular layers. System of (3.2) with $\varepsilon = 0$ is the *limiting fast system*, whose orbits are called *fast orbits* or singular (boundary and/or internal) layers. In this context, *a singular orbit* of system (3.1) or (3.2) is defined to be a continuous and piecewise smooth curve in $\mathbb{R}^7$ that is a union of finitely many slow and fast orbits. Very often, limiting slow and fast systems provide complementary information on state variables. Therefore, the main task of singularly perturbed problems is to patch the limiting information together to form a solution for the entire $\varepsilon > 0$ system.

Let B_L and B_R be the subsets of the phase space $\mathbb{R}^7$ defined by

$$\begin{aligned} B_L =& \{(V, u, L_1, L_2, J_1, J_2, 0) \in \mathbb{R}^7 : \text{arbitrary } u, J_1, J_2\}, \\ B_R =& \{(0, u, R_1, R_2, J_1, J_2, 1) \in \mathbb{R}^7 : \text{arbitrary } u, J_1, J_2\}. \end{aligned} \tag{3.3}$$

Then the original BVP is equivalent to the connecting problem, namely, *finding an orbit of (3.1) or (3.2) from B_L to B_R* (see, for example, [44]).

In what follows, we will consider the equivalent connecting problem for system (3.1) or (3.2). The construction of a connecting orbit involves two main steps ([12, 44, 45, 46, 52, 54, 78, 79, 85]):

Step I: We construct a singular orbit to the connecting problem.

Step II: We apply geometric singular perturbation theory, particularly, the Exchange Lemmas, to show that there is a unique connecting orbit near the singular orbit for small $\varepsilon > 0$.

3.1. **Geometric construction of singular orbits.** Following the idea in [20, 53, 55], we will first construct a singular orbit on $[0, 1]$ that connects B_L to B_R. Such an orbit will generally consist of two boundary layers and a regular layer.

3.1.1. *Limiting fast dynamics and boundary layers.* By setting $\varepsilon = 0$ in (3.1), we obtain the *slow manifold*

$$\mathcal{Z} = \{u = 0,\ z_1c_1 + z_2c_2 = 0\}.$$

By setting $\varepsilon = 0$ in (3.2), we get the *limiting fast system*

$$\begin{aligned} \phi' =& u, \quad u' = -z_1c_1 - z_2c_2, \quad c_1' = -f_1(c_1, c_2; \nu_1, \nu_2)u, \\ c_2' =& f_2(c_1, c_2; \nu_1, \nu_2)u, \quad J_1' = J_2' = 0, \quad \tau' = 0. \end{aligned} \tag{3.4}$$

Note that the slow manifold $\mathcal{Z}$ is the set of equilibria of (3.4).

Lemma 3.1. *For system (3.4), the slow manifold $\mathcal{Z}$ is normally hyperbolic.*

Proof. Even though the LHS models used in [51] and in this paper are different, the proof of this result is the same, word by word, as that in [51]. For convenience of the reader, we provide the key ingredients here.

The linearization of (3.4) at each point of $(\phi, 0, c_1, c_2, J_1, J_2, \tau) \in \mathcal{Z}$ has five zero eigenvalues associated with the set of equilibria $\mathcal{Z}$ with $\dim(\mathcal{Z}) = 5$, and the other two eigenvalues are

$$\pm\sqrt{z_1f_1 - z_2f_2} = \pm\sqrt{z_1^2c_1 + z_2^2c_2}.$$

Note that $f_1(c_1, c_2; \nu_1, \nu_2)$ has a factor c_1 and $f_2(c_1, c_2; \nu_1, \nu_2)$ has a factor c_2. It follows from (c_1, c_2)-subsystem of (3.4) that $\{c_1 > 0\}$ and $\{c_2 > 0\}$ are invariant. Since c_1 and c_2 have positive boundary values, c_1 and c_2 are positive for all $x \in [0, 1]$. Therefore, $z_1f_1 - z_2f_2 > 0$. Thus $\mathcal{Z}$ is normally hyperbolic. □

We denote the stable (resp. unstable) manifold of $\mathcal{Z}$ by $W^s(\mathcal{Z})$ (resp. $W^u(\mathcal{Z})$). Let M_L be the collection of orbits from B_L in forward time under the flow of system (3.4) and let M_R be the collection of orbits from B_R in backward time under the flow of system (3.4). Then, for a singular orbit connecting B_L to B_R, the boundary layer at $x = 0$ must lie in $N_L = M_L \cap W^s(\mathcal{Z})$ and the boundary layer at $x = 1$ must lie in $N_R = M_R \cap W^u(\mathcal{Z})$. In this subsection, we will determine the boundary layers N_L and N_R, and their landing points $\omega(N_L)$ and $\alpha(N_R)$ on the slow manifold

$\mathcal{Z}$. The regular layer, determined by the limiting slow system in §3.1.2, will lie in $\mathcal{Z}$ and connect $\omega(N_L)$ at $x=0$ and $\alpha(N_R)$ at $x=1$.

Recall the definitions of ν_1 and ν_2 from (2.7) and the discussion in the last paragraph of Section 2. We will be interested in the situation that ν_1 and ν_2 are small and treat (3.4) as a regular perturbation of that with $\nu_1=\nu_2=0$. While ν_1 and ν_2 are small, their ratio is of order $O(1)$. We thus set

$$\nu_1=\nu \ \text{ and } \ \nu_2=\lambda\nu \tag{3.5}$$

and look for solutions

$$\Gamma(\xi;\nu)=(\phi(\xi;\nu),u(\xi;\nu),c_1(\xi;\nu),c_2(\xi;\nu),J_1(\nu),J_2(\nu),\tau)$$

of system (3.4) of the form

$$\begin{aligned}&\phi(\xi;\nu)=\phi_0(\xi)+\phi_1(\xi)\nu+o(\nu), \quad u(\xi;\nu)=u_0(\xi)+u_1(\xi)\nu+o(\nu),\\ &c_i(\xi;d)=c_{i0}(\xi)+c_{i1}(\xi)\nu+o(\nu), \quad J_i(\nu)=J_{i0}+J_{i1}\nu+o(\nu).\end{aligned} \tag{3.6}$$

Substituting (3.6) into system (3.4), we obtain, for the zeroth order in ν,

$$\begin{aligned}\phi_0'=&u_0, \quad u_0'=-z_1c_{10}-z_2c_{20}, \quad c_{10}'=-z_1c_{10}u_0,\\ c_{20}'=&-z_2c_{20}u_0, \quad J_{10}'=J_{20}'=0, \quad \tau'=0,\end{aligned} \tag{3.7}$$

and, for the first order in ν,

$$\begin{aligned}\phi_1'=&u_1, \quad u_1'=-z_1c_{11}-z_2c_{21},\\ c_{11}'=&-z_1u_0c_{11}-z_1c_{10}u_1+(z_1c_{10}+\lambda z_2c_{20})c_{10}u_0,\\ c_{21}'=&-z_2u_0c_{21}-z_2c_{20}u_1+(z_1c_{10}+\lambda z_2c_{20})c_{20}u_0,\\ J_{11}'=&J_{21}'=0, \quad \tau'=0.\end{aligned} \tag{3.8}$$

Recall that we are interested in $\Gamma^0(\xi;\nu)\subset N_L=M_L\cap W^s(\mathcal{Z})$ with $\Gamma^0(0;\nu)\in B_L$ and $\Gamma^1(\xi;\nu)\subset N_R=M_R\cap W^u(\mathcal{Z})$ with $\Gamma^1(0;\nu)\in B_R$.

Proposition 3.2. *Assume $\nu\ge 0$ is small.*

(i) *The stable manifold $W^s(\mathcal{Z})$ intersects B_L transversally at points*

$$\left(V,u_0^l+u_1^l\nu+o(\nu),L_1,L_2,J_1(\nu),J_2(\nu),0\right),$$

and the ω-limit set of $N_L=M_L\bigcap W^s(\mathcal{Z})$ is

$$\omega(N_L)=\left\{(\phi_0^L+\phi_1^L\nu+o(\nu),0,c_{10}^L+c_{11}^L\nu+o(\nu),c_{20}^L+c_{21}^L\nu+o(\nu),J_1(\nu),J_2(\nu),0)\right\},$$

where $J_i(\nu)=J_{i0}+J_{i1}\nu+o(\nu)$, $i=1,2$, can be arbitrary and

$$\begin{aligned}\phi_0^L=&V-\frac{1}{z_1-z_2}\ln\frac{-z_2L_2}{z_1L_1}, \quad z_1c_{10}^L=-z_2c_{20}^L=(z_1L_1)^{\frac{-z_2}{z_1-z_2}}(-z_2L_2)^{\frac{z_1}{z_1-z_2}},\\ u_0^l=&sgn(z_1L_1+z_2L_2)\sqrt{2\left(L_1+L_2+\frac{z_1-z_2}{z_1z_2}(z_1L_1)^{\frac{-z_2}{z_1-z_2}}(-z_2L_2)^{\frac{z_1}{z_1-z_2}}\right)};\\ \phi_1^L=&0, \quad z_1c_{11}^L=-z_2c_{21}^L=z_1c_{10}^L(L_1+\lambda L_2-c_{10}^L-\lambda c_{20}^L),\\ u_1^l=&\frac{1}{u_0^L}\left(\frac{\lambda}{2}(L_2+c_{20}^L)(L_2-c_{20}^L)+\frac{1}{2}(L_1+c_{10}^L)(L_1-c_{10}^L)-c_{10}^Lc_{20}^L-c_{11}^L-c_{21}^L\right.\\ &\left.-\frac{z_2(1-\lambda)}{z_1+z_2}e^{(z_1+z_2)(V-\phi_0^L)}\right).\end{aligned}$$

(ii) *The unstable manifold* $W^u(\mathcal{Z})$ *intersects* B_R *transversally at points*

$$(0, u_0^r + u_1^r\nu + o(\nu), R_1, R_2, J_1(\nu), J_2(\nu), 1),$$

and the α*-limit set of* N_R *is*

$$\alpha(N_R) = \left\{(\phi_0^R + \phi_1^R\nu + o(\nu), 0, c_{10}^R + c_{11}^R\nu + o(\nu), c_{20}^R + c_{21}^R\nu + o(\nu), J_1(\nu), J_2(\nu), 1)\right\},$$

where $J_i(\nu) = J_{i0} + J_{i1}\nu + o(\nu)$*,* $i = 1, 2$*, can be arbitrary and*

$$\phi_0^R = -\frac{1}{z_1 - z_2}\ln\frac{-z_2R_2}{z_1R_1}, \quad z_1c_{10}^R = -z_2c_{20}^R = (z_1R_1)^{\frac{-z_2}{z_1-z_2}}(-z_2R_2)^{\frac{z_1}{z_1-z_2}},$$

$$u_0^r = -\,sgn(z_1R_1 + z_2R_2)\sqrt{2\left(R_1 + R_2 + \frac{z_1 - z_2}{z_1z_2}(z_1R_1)^{\frac{-z_2}{z_1-z_2}}(-z_2R_2)^{\frac{z_1}{z_1-z_2}}\right)};$$

$$\phi_1^R = 0, \quad z_1c_{11}^R = -z_2c_{21}^R = z_1c_{10}^R(R_1 + \lambda R_2 - c_{10}^R - \lambda c_{20}^R),$$

$$u_1^r = \frac{1}{u_0^r}\left(\frac{\lambda}{2}(R_2 + c_{20}^R)(R_2 - c_{20}^R) + \frac{1}{2}(R_1 + c_{10}^R)(R_1 - c_{10}^R) - c_{10}^Rc_{20}^R - c_{11}^R - c_{21}^R - \frac{z_2(1-\lambda)}{z_1 + z_2}e^{-(z_1+z_2)\phi_0^R}\right).$$

Proof. The stated result for system (3.7) has been obtained in [20, 53, 55]. For system (3.8), one can check directly that it has three nontrivial first integrals:

$$F_1 = \frac{c_{11}}{c_{10}} + z_1\phi_1 + c_{10} + \lambda c_{20}, \quad F_2 = \frac{c_{21}}{c_{20}} + z_2\phi_1 + c_{10} + \lambda c_{20},$$

$$F_3 = u_0u_1 - c_{11} - c_{21} - \frac{\lambda}{2}c_{20}^2 - \frac{1}{2}c_{10}^2 - c_{10}c_{20} + \frac{z_2(1-\lambda)}{z_1 + z_2}e^{(z_1+z_2)(V-\phi_0)}.$$

We now establish the results for $\phi_1^L, c_{11}^L, c_{21}^L$ and u_1^l for system (3.8). Those for $\phi_1^R, c_{11}^R, c_{21}^R$ and u_1^r can be established in the similar way.

Note that $\phi_1(0) = c_{11}(0) = c_{21}(0) = 0$. Using the integrals F_1 and F_2, we have

$$\frac{c_{11}}{c_{10}} + z_1\phi_1 + c_{10} + \lambda c_{20} = L_1 + \lambda L_2, \text{ and } \frac{c_{21}}{c_{20}} + z_2\phi_1 + c_{10} + \lambda c_{20} = L_1 + \lambda L_2.$$

Therefore

$$c_{11} = c_{10}(L_1 + \lambda L_2 - c_{10} - \lambda c_{20} - z_1\phi_1), \quad c_{21} = c_{20}(L_1 + \lambda L_2 - c_{10} - \lambda c_{20} - z_2\phi_1).$$

Taking the limit as $\xi \to \infty$, we have

$$\phi_1^L = 0, \quad c_{11}^L = c_{10}^L(L_1 + \lambda L_2 - c_{10}^L - \lambda c_{20}^L), \quad c_{21}^L = c_{20}^L(L_1 + \lambda L_2 - c_{10}^L - \lambda c_{20}^L).$$

In view of the relations $z_1c_{10}^L + z_2c_{20}^L = z_1c_{11}^L + z_2c_{21}^L = 0$, one can get the formulas for c_{11}^L, c_{21}^L and ϕ_1^L. We now derive the formula for $u_1^l = u_1(0)$.

In view of $F_3(0) = F_3(\infty)$, we have

$$u_0^lu_1^l - L_1L_2 - \frac{\lambda}{2}L_2^2 - \frac{1}{2}L_1^2 = -c_{11}^L - c_{21}^L - c_{10}^Lc_{20}^L - \frac{\lambda}{2}\left(c_{10}^L\right)^2 - \frac{1}{2}\left(c_{10}^L\right)^2 - \frac{z_2(1-\lambda)}{z_1 + z_2}e^{(z_1+z_2)(V-\phi_0^L)}.$$

The formula for u_1^l follows directly. This completes the proof. □

We remark that, when $z_1L_1 + z_2L_2 = 0$, $u_0^l = 0$. In this case, u_1^l is defined as the limit of its expression as $z_1L_1 + z_2L_2 \to 0$ and it is zero. Similar remark applies to u_1^r when $z_1R_1 + z_2R_2 = 0$.

For later use, let Γ^0 denote the possible boundary layer at $x = 0$ and let Γ^1 denote the possible boundary layer at $x = 1$ for system (3.4).

Corollary 3.3. *Under electroneutrality boundary conditions, that is,*

$$z_1L_1 = -z_2L_2 = L \quad \textit{and} \quad z_1R_1 = -z_2R_2 = R, \tag{3.9}$$

one has

$$\phi_0^L = V,\ z_1c_{10}^L = -z_2c_{20}^L = L;\ \phi_0^R = 0,\ z_1c_{10}^R = -z_2c_{20}^R = R;$$
$$\phi_1^L = c_{11}^L = c_{21}^L = \phi_1^R = c_{11}^R = c_{21}^R = 0.$$

In particular, up to $O(\nu)$, there is no boundary layer at $x = 0$ and $x = 1$.

3.1.2. *Limiting slow dynamics and regular layer.* Next we construct the regular layer on $\mathcal{Z}$ that connects $\omega(N_L)$ and $\alpha(N_R)$. Note that, for $\varepsilon = 0$, system (3.1) loses most information. To remedy this degeneracy, we follow the idea in [20, 53, 55] and make a rescaling $u = \varepsilon p$ and $-z_2c_2 = z_1c_1 + \varepsilon q$ in system (3.1). In term of the new variables, system (3.1) becomes

$$\begin{aligned}\dot\phi =& p, \quad \varepsilon\dot p = q - \varepsilon\frac{h_\tau(\tau)}{h(\tau)}p, \quad \varepsilon\dot q = (z_1f_1 - z_2f_2)\,p + \frac{z_1g_1 + z_2g_2}{h(\tau)},\\ \dot c_1 =& -f_1p - \frac{g_1}{h(\tau)}, \quad \dot J_1 = \dot J_2 = 0, \quad \dot\tau = 1\end{aligned} \tag{3.10}$$

where, for $i = 1, 2$,

$$f_i = f_i\left(c_1, -\frac{z_1c_1 + \varepsilon q}{z_2}; \nu, \lambda\nu\right);\ g_1 = g_1\left(c_1, J_1, J_2; \nu, \lambda\nu\right);$$

and

$$g_2 = g_2\left(-\frac{z_1c_1 + \varepsilon q}{z_2}, J_1, J_2; \nu, \lambda\nu\right).$$

It is again a singular perturbation problem and its limiting slow system is

$$\begin{aligned}\dot\phi =& p, \quad q = 0, \quad p = -\frac{z_1g_1\big(c_1, J_1, J_2; \nu, \lambda\nu\big) + z_2g_2\big(-\frac{z_1}{z_2}c_1, J_1, J_2; \nu, \lambda\nu\big)}{z_1(z_1 - z_2)h(\tau)c_1},\\ \dot c_1 =& -f_1\big(c_1, -\frac{z_1}{z_2}c_1; \nu, \lambda\nu\big)p - \frac{1}{h(\tau)}g_1\big(c_1, J_1, J_2; \nu, \lambda\nu\big),\\ \dot J_1 =& \dot J_2 = 0, \quad \dot\tau = 1.\end{aligned} \tag{3.11}$$

For system (3.11), the slow manifold is

$$\mathcal{S} = \left\{q = 0,\ p = -\frac{z_1g_1\big(c_1, J_1, J_2; \nu, \lambda\nu\big) + z_2g_2\big(-\frac{z_1}{z_2}c_1, J_1, J_2; \nu, \lambda\nu\big)}{z_1(z_1 - z_2)h(\tau)c_1}\right\}.$$

Therefore, the limiting slow system on $\mathcal{S}$ is

$$\begin{aligned}\dot\phi =& p, \quad \dot c_1 = -f_1\big(c_1, -\frac{z_1}{z_2}c_1; \nu, \lambda\nu\big)p - \frac{1}{h(\tau)}g_1\big(c_1, J_1, J_2; \nu, \lambda\nu\big),\\ \dot J_1 =& \dot J_2 = 0, \quad \dot\tau = 1,\end{aligned} \tag{3.12}$$

where

$$p = -\frac{z_1g_1\big(c_1, J_1, J_2; \nu, \lambda\nu\big) + z_2g_2\big(-\frac{z_1}{z_2}c_1, J_1, J_2; \nu, \lambda\nu\big)}{z_1(z_1 - z_2)h(\tau)c_1}.$$

Similar to the layer problem, we look for solutions of (3.12) of the form

$$\begin{aligned}&\phi(x) = \phi_0(x) + \phi_1(x)\nu + o(\nu), \quad c_1(x) = c_{10}(x) + c_{11}(x)\nu + o(\nu),\\ &J_i = J_{i0} + J_{i1}\nu + o(\nu).\end{aligned} \tag{3.13}$$

to connect $\omega(N_L)$ and $\alpha(N_R)$ given in Proposition 3.2; in particular, for $j=0,1$,

$$(\phi_j(0), c_{1j}(0)) = \left(\phi_j^L, c_{1j}^L\right), \quad (\phi_j(1), c_{1j}(1)) = \left(\phi_j^R, c_{1j}^R\right).$$

From system (3.12) and the definitions of f_j's and g_j's in (2.10), we have

$$\begin{aligned}\dot\phi_0 &= -\frac{z_1J_{10}+z_2J_{20}}{z_1(z_1-z_2)h(\tau)c_{10}}, \quad \dot c_{10} = \frac{z_2\left(J_{10}+J_{20}\right)}{(z_1-z_2)h(\tau)},\\ \dot J_{10} &= \dot J_{20} = 0, \quad \dot\tau = 1,\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\dot\phi_1 =& \frac{(z_1J_{10}+z_2J_{20})c_{11}}{z_1(z_1-z_2)h(\tau)c_{10}^2} - \frac{z_1J_{11}+z_2J_{21}}{z_1(z_1-z_2)h(\tau)c_{10}},\\ \dot c_{11} =& \frac{(z_1\lambda-z_2)(J_{10}+J_{20})c_{10}}{(z_1-z_2)h(\tau)} + \frac{z_2(J_{11}+J_{21})}{(z_1-z_2)h(\tau)},\\ \dot J_{11} =& \dot J_{21} = 0, \quad \dot\tau = 1.\end{aligned} \tag{3.15}$$

We denote

$$H(x) = \int_0^x h^{-1}(s)ds. \tag{3.16}$$

Lemma 3.4. *There is a unique solution $(\phi_0(x), c_{10}(x), J_{10}, J_{20}, \tau(x))$ of (3.14) such that*

$$(\phi_0(0), c_{10}(0), \tau(0)) = (\phi_0^L, c_{10}^L, 0) \ \ \text{and} \ \ (\phi_0(1), c_{10}(1), \tau(1)) = (\phi_0^R, c_{10}^R, 1),$$

where ϕ_0^L, ϕ_0^R, c_{10}^L, and c_{10}^R are given in Proposition 3.2. It is given by

$$\begin{aligned}\phi_0(x) =& \phi_0^L + \frac{\phi_0^R-\phi_0^L}{\ln c_{10}^R - \ln c_{10}^L}\ln\left(1-\frac{H(x)}{H(1)}+\frac{H(x)}{H(1)}\frac{c_{10}^R}{c_{10}^L}\right),\\ c_{10}(x) =& \left(1-\frac{H(x)}{H(1)}\right)c_{10}^L + \frac{H(x)}{H(1)}c_{10}^R,\\ J_{10} =& \frac{c_{10}^L-c_{10}^R}{H(1)(\ln c_{10}^L-\ln c_{10}^R)}\left(z_1\left(\phi_0^L-\phi_0^R\right)+\ln c_{10}^L-\ln c_{10}^R\right),\\ J_{20} =& \frac{c_{20}^L-c_{20}^R}{H(1)(\ln c_{20}^L-\ln c_{20}^R)}\left(z_2\left(\phi_0^L-\phi_0^R\right)+\ln c_{20}^L-\ln c_{20}^R\right),\\ \tau(x) =& x.\end{aligned} \tag{3.17}$$

Proof. We refer the readers to [20, 53, 55] for a detailed proof. □

We now examine system (3.15). For convenience, we define two functions

$$M = M(L_1, L_2, R_1, R_2; \lambda), \ N = N(L_1, L_2, R_1, R_2; \lambda)$$

as

$$\begin{aligned}M =& c_{11}^R - c_{11}^L + \frac{z_1\lambda-z_2}{2z_2}(c_{10}^L+c_{10}^R)(c_{10}^L-c_{10}^R),\\ N =& \frac{\phi_0^L-\phi_0^R}{\ln c_{10}^L-\ln c_{10}^R}\left(\frac{c_{11}^R}{c_{10}^R}-\frac{c_{11}^L}{c_{10}^L}+\frac{z_1\lambda-z_2}{z_2}(c_{10}^L-c_{10}^R)\right) - \frac{\phi_0^L-\phi_0^R}{c_{10}^L-c_{10}^R}M.\end{aligned} \tag{3.18}$$

Lemma 3.5. *There is a unique solution $(\phi_1(x), c_{11}(x), J_{11}, J_{21}, \tau(x))$ of (3.15) such that*

$$(\phi_1(0), c_{11}(0), \tau(0)) = (\phi_1^L, c_{11}^L, 0) \ \ \text{and} \ \ (\phi_1(1), c_{11}(1), \tau(1)) = (\phi_1^R, c_{11}^R, 1),$$

where ϕ_1^L, ϕ_1^R, c_{11}^L, and c_{11}^R are given in Proposition 3.2. It is given by

$$\begin{aligned}\phi_1(x) =& -\frac{\ln c_{10}^L - \ln c_{10}(x)}{\ln c_{10}^L - \ln c_{10}^R}N + \frac{\phi_0^L - \phi_0^R}{\ln c_{10}^L - \ln c_{10}^R}\left[\frac{c_{10}^L - c_{10}(x)}{c_{10}(x)}\left(\frac{c_{11}^L}{c_{10}^L} - \frac{z_1\lambda - z_2}{2z_2}c_{10}^L\right)\right.\\ &\left. + \frac{(z_1\lambda - z_2)(c_{10}^L - c_{10}^R)}{2z_2}\frac{H(x)}{H(1)} + \left(\frac{H(x)}{H(1)c_{10}(x)} - \frac{\ln c_{10}^L - \ln c_{10}(x)}{c_{10}^L - c_{10}^R}\right)M\right],\\ c_{11}(x) =& c_{11}^L - \frac{z_1\lambda - z_2}{2z_2}\left(c_{10}^L + c_{10}(x)\right)\left(c_{10}^L - c_{10}(x)\right) + \frac{H(x)}{H(1)}M,\\ J_{11} =& \frac{z_1(c_{10}^L - c_{10}^R)}{H(1)(\ln c_{10}^L - \ln c_{10}^R)}N - \frac{M}{H(1)}, \quad J_{21} = -\frac{z_1(c_{10}^L - c_{10}^R)}{H(1)(\ln c_{10}^L - \ln c_{10}^R)}N + \frac{z_1 M}{z_2 H(1)},\end{aligned}$$

where M and N are defined in (3.18).

Proof. It follows from (3.15) that

$$\begin{aligned}c_{11}(x) =& c_{11}^L + \frac{(z_1\lambda - z_2)(J_{10} + J_{20})}{z_1 - z_2}\int_0^x \frac{c_{10}(s)}{h(s)}ds + \frac{z_2(J_{11} + J_{21})}{z_1 - z_2}\int_0^x \frac{1}{h(s)}ds\\ =& c_{11}^L + \frac{z_1\lambda - z_2}{2z_2}\left(c_{10}^2(x) - \left(c_{10}^L\right)^2\right) + \frac{z_2(J_{11} + J_{21})}{z_1 - z_2}H(x).\end{aligned}$$

Thus, from Proposition 3.2,

$$\frac{z_2(J_{11} + J_{21})}{z_1 - z_2}H(1) = c_{11}^R - c_{11}^L + \frac{z_1\lambda - z_2}{2z_2}\left(c_{10}^L + c_{10}^R\right)\left(c_{10}^L - c_{10}^R\right),$$

which gives, by definition of M in (3.18),

$$J_{11} + J_{21} = \frac{z_1 - z_2}{z_2 H(1)}M. \tag{3.19}$$

Hence,

$$c_{11}(x) = c_{11}^L - \frac{z_1\lambda - z_2}{2z_2}\left(c_{10}^L + c_{10}(x)\right)\left(c_{10}^L - c_{10}(x)\right) + \frac{H(x)}{H(1)}M. \tag{3.20}$$

Also, from (3.15), we have

$$\phi_1(x) = \phi_1^L + \frac{z_1 J_{10} + z_2 J_{20}}{z_1(z_1 - z_2)}\int_0^x \frac{c_{11}(s)}{h(s)c_{10}^2(s)}ds - \frac{z_1 J_{11} + z_2 J_{21}}{z_1(z_1 - z_2)}\int_0^x \frac{1}{h(s)c_{10}(s)}ds.$$

Note that, from (3.14) and (3.17),

$$\begin{aligned}\int_0^x \frac{1}{h(s)c_{10}^2(s)}ds =& \frac{z_1 - z_2}{z_2(J_{10} + J_{20})}\int_0^x \frac{\dot{c}_{10}(s)}{c_{10}^2(s)}ds = \frac{H(1)(c_{10}^L - c_{10}(x))}{(c_{10}^L - c_{10}^R)c_{10}^L c_{10}(x)},\\ \int_0^x \frac{\int_0^s h^{-1}(\sigma)d\sigma}{h(s)c_{10}^2(s)}ds =& -\frac{z_1 - z_2}{z_2(J_{10} + J_{20})}\int_0^x\int_0^s h^{-1}(\sigma)d\sigma\frac{d}{ds}c_{10}^{-1}(s)ds\\ =& \frac{H(1)}{c_{10}^L - c_{10}^R}\left(\frac{H(x)}{c_{10}(x)} - \int_0^x h^{-1}(s)c_{10}^{-1}(s)ds\right)\\ =& \frac{H(1)H(x)}{(c_{10}^L - c_{10}^R)c_{10}(x)} - \frac{H(1)^2(\ln c_{10}^L - \ln c_{10}(x))}{(c_{10}^L - c_{10}^R)^2}.\end{aligned}$$

These, together with (3.20) and (3.17), yield

$$\int_0^x \frac{c_{11}(s)}{h(s)c_{10}^2(s)}ds = \left(c_{11}^L - \frac{z_1\lambda - z_2}{2z_2}\left(c_{10}^L\right)^2\right)\frac{\left(c_{10}^L - c_{10}(x)\right)H(1)}{(c_{10}^L - c_{10}^R)c_{10}^L c_{10}(x)} + \frac{z_1\lambda - z_2}{2z_2}H(x)$$
$$+ \frac{M}{c_{10}^L - c_{10}^R}\left(\frac{H(x)}{c_{10}(x)} - \frac{\ln c_{10}^L - \ln c_{10}(x)}{c_{10}^L - c_{10}^R}\right).$$

A direct calculation then gives

$$\phi_1(x) = -\frac{\ln c_{10}^L - \ln c_{10}(x)}{\ln c_{10}^L - \ln c_{10}^R}N + \frac{\phi_0^L - \phi_0^R}{\ln c_{10}^L - \ln c_{10}^R}\left[\frac{c_{10}^L - c_{10}(x)}{c_{10}(x)}\left(\frac{c_{11}^L}{c_{10}^L} - \frac{z_1\lambda - z_2}{2z_2}c_{10}^L\right)\right.$$
$$\left. + \frac{(z_1\lambda - z_2)(c_{10}^L - c_{10}^R)}{2z_2}\frac{H(x)}{H(1)} + \left(\frac{H(x)}{H(1)c_{10}(x)} - \frac{\ln c_{10}^L - \ln c_{10}(x)}{c_{10}^L - c_{10}^R}\right)M\right].$$

Hence,

$$\frac{(z_1J_{11} + z_2J_{21})(\ln c_{10}^L - \ln c_{10}^R)}{z_1(z_1 - z_2)(c_{10}^L - c_{10}^R)}H(1) - N = 0,$$

which gives

$$z_1J_{11} + z_2J_{21} = \frac{z_1(z_1 - z_2)(c_{10}^L - c_{10}^R)}{H(1)(\ln c_{10}^L - \ln c_{10}^R)}N.$$

Formulas for J_{11}, J_{21}, and ϕ_1 follow directly. □

The slow orbit

$$\Lambda(x;\nu) = (\phi_0(x) + \nu\phi_1(x), c_{10}(x) + \nu c_{11}(x), J_1(\nu), J_2(\nu), \tau(x)) + o(\nu) \tag{3.21}$$

given in Lemmas 3.4 and 3.5 connects $\omega(N_L)$ and $\alpha(N_R)$. Let $\bar{M}_L$ (resp., $\bar{M}_R$) be the forward (resp., backward) image of $\omega(N_L)$ (resp., $\alpha(N_R)$) under the slow flow (3.12). One has the following result.

Proposition 3.6. *There exists $\nu_0 > 0$ small depending on boundary conditions so that, if $0 \le \nu \le \nu_0$, then, on the five-dimensional slow manifold $\mathcal{S}$, $\bar{M}_L$ and $\bar{M}_R$ intersects transversally along the unique orbit $\Lambda(x;\nu)$ given in (3.21).*

Proof. We will establish the transversality of the intersection by showing that $\omega(N_L)\cdot 1$ (the image of $\omega(N_L)$ under the time-one map of the flow of system (3.12)) is transversal to $\alpha(N_R)$ on $\mathcal{S}\bigcap\{\tau = 1\}$. It consists of the following two steps.

Step 1: We will show that, for $\nu = 0$, $\omega(N_L)\cdot 1$ and $\alpha(N_R)$ intersect transversally on $\mathcal{S}\bigcap\{\tau = 1\}$.

Using (ϕ, c_1, J_1, J_2) as a coordinate system on $\mathcal{S}\bigcap\{\tau = 1\}$, it then follows from (3.17) that, for $\nu = 0$, $\omega(N_L)\cdot 1$ is given by

$$\omega(N_L)\cdot 1 = \{(\phi(J_1, J_2), c_1(J_1, J_2), J_1, J_2) : \text{ arbitrary } J_1,\ J_2\}$$

with

$$\phi(J_1, J_2) = \phi_0^L - \frac{z_1J_1 + z_2J_2}{z_1z_2(J_1 + J_2)}\ln\frac{c_1(J_1, J_2)}{c_{10}^L}, \quad c_1(J_1, J_2) = c_{10}^L + \frac{z_2H(1)(J_1 + J_2)}{z_1 - z_2}.$$

Therefore, the tangent space to $\omega(N_L)\cdot 1$ restricted on $\mathcal{S}\bigcap\{\tau = 1\}$ is spanned by the vectors

$$(\phi_{J_1}, (c_1)_{J_1}, 1, 0) = \left(\phi_{J_1}, \frac{z_2H(1)}{z_1 - z_2}, 1, 0\right) \text{ and } (\phi_{J_2}, (c_1)_{J_2}, 0, 1) = \left(\phi_{J_2}, \frac{z_2H(1)}{z_1 - z_2}, 0, 1\right).$$

In view of the display in Proposition 3.2, the set $\alpha(N_R)$ is parameterized by J_1 and J_2, and hence, the tangent space to $\alpha(N_R)$ restricted on $\mathcal{S}\bigcap\{\tau=1\}$ is spanned by (0,0,1,0) and (0,0,0,1). Note that $\mathcal{S}\bigcap\{\tau=1\}$ is four dimensional. Thus, it suffices to show that the above four vectors are linearly independent or, equivalently, $\phi_{J_1}\neq\phi_{J_2}$ at $(J_1,J_2)=(J_{10},J_{20})$. The latter can be verified by a direct computation as follows:

$$\phi_{J_1}-\phi_{J_2}=-\frac{z_1-z_2}{z_1z_2(J_1+J_2)}\ln\left[1+\frac{z_2(J_1+J_2)}{(z_1-z_2)c_{10}^L}H(1)\right]\neq 0,$$

even as $J_1+J_2\to 0$. This establishes the transversal intersection of $\omega(N_L)\cdot 1$ and $\alpha(N_R)$ on $\mathcal{S}\bigcap\{\tau=1\}$.

Step 2: We show that there exists $\nu_0>0$ small, so that, if $0\leq\nu\leq\nu_0$, then $\omega(N_L)\cdot 1$ and $\alpha(N_R)$ intersect transversally on $\mathcal{S}\bigcap\{\tau=1\}$.

This can be argued directly from the smooth dependence of solutions on parameter ν. And we complete the proof. $\square$

3.2. **Existence of solutions near the singular orbit.** We have constructed a unique singular orbit on [0,1] that connects B_L to B_R. It consists of two boundary layer orbits Γ^0 from the point

$$(V,u_0^l+u_1^l\nu+o(\nu),L_1,L_2,J_{10}+J_{11}\nu+o(\nu),J_{20}+J_{21}\nu+o(\nu),0)\in B_L$$

to the point

$$(\phi^L,0,c_1^L,c_2^L,J_1,J_2,0)\in\omega(N_L)\subset\mathcal{Z}$$

and Γ^1 from the point

$$(\phi^R,0,c_1^R,c_2^R,J_1,J_2,1)\in\alpha(N_R)\subset\mathcal{Z}$$

to the point

$$(0,u_0^r+u_1^r+o(\nu),R_1,R_2,J_{10}+J_{11}\nu+o(\nu),J_{20}+J_{21}\nu+o(\nu),1)\in B_R,$$

and a regular layer Λ on $\mathcal{Z}$ that connects the two landing points

$$(\phi^L,0,c_1^L,c_2^L,J_1,J_2,0)\in\omega(N_L)\quad\text{and}\quad(\phi^R,0,c_1^R,c_2^R,J_1,J_2,1)\in\alpha(N_R)$$

of the two boundary layers.

We now establish the existence of a solution of (2.8)-(2.9) near the singular orbit constructed above which is a union of two boundary layers and one regular layer $\Gamma^0\cup\Lambda\cup\Gamma^1$. The proof follows the same line as that in [20, 51, 53, 55], and the main tool used is the Exchange Lemma (see, for example, [12, 44, 45, 46, 52, 54, 78, 79, 85]) of the geometric singular perturbation theory.

Theorem 3.7. *Let $\Gamma^0\cup\Lambda\cup\Gamma^1$ be the singular orbit of the connecting problem system (3.1) associated to B_L and B_R in system (3.3). Then, for $\varepsilon>0$ small and $\nu\geq 0$ small, the boundary value problem (2.8) and (2.9) has a unique smooth solution near the singular orbit.*

Proof. Let $\nu_0>0$ be as in Proposition 3.6. For $0\leq\nu\leq\nu_0$, denote $u^l=u_0^l+u_1^l\nu$, $J_1(\nu)=J_{10}+J_{11}\nu$ and $J_2(\nu)=J_{20}+J_{21}\nu$. Fix $\delta>0$ small to be determined. Let

$$B_L(\delta)=\{(V,u,L_1,L_2,J_1,J_2,0)\in\mathbb{R}^7:|u-u^l|<\delta,|J_i-J_i(\nu)|<\delta\}.$$

For $\varepsilon>0$, let $M_L(\varepsilon,\delta)$ be the forward trace of $B_L(\delta)$ under the flow of system (3.1) or equivalently of system (3.2) and let $M_R(\varepsilon)$ be the backward trace of B_R. To prove the existence and uniqueness statement, it suffices to show that $M_L(\varepsilon,\delta)$

intersects $M_R(\varepsilon)$ transversally in a neighborhood of the singular orbit $\Gamma^0 \cup \Lambda \cup \Gamma^1$. The latter will be established by an application of Exchange Lemmas.

Note that $\dim B_L(\delta)$=3. It is clear that the vector field of the fast system (3.2) is not tangent to $B_L(\delta)$ for $\varepsilon \geq 0$, and hence, $\dim M_L(\varepsilon,\delta)$=4. We next apply Exchange Lemma to track $M_L(\varepsilon,\delta)$ in the vicinity of $\Gamma^0 \cup \Lambda \cup \Gamma^1$. First of all, the transversality of the intersection $B_L(\delta)\bigcap W^s(\mathcal{Z})$ along Γ^0 in Proposition 3.2 implies the transversality of intersection $M_L(0,\delta)\bigcap W^s(\mathcal{Z})$. Secondly, we have also established that $\dim \omega(N_L) = \dim N_L - 1 = 2$ in Proposition 3.2 and that the limiting slow flow is not tangent to $\omega(N_L)$ in Section 3.1.2. With these conditions, Exchange Lemma ([12, 44, 45, 46, 78, 85]) states that there exist $\rho > 0$ and $\varepsilon_1 > 0$ so that, if $0 < \varepsilon \leq \varepsilon_1$, then $M_L(\varepsilon,\delta)$ will first follow Γ^0 toward $\omega(N_L) \subset \mathcal{Z}$, then follow the trace of $\omega(N_L)$ in the vicinity of Λ toward $\{\tau = 1\}$, leave the vicinity of $\mathcal{Z}$, and, upon exit, a portion of $M_L(\varepsilon,\delta)$ is C^1 $O(\varepsilon)$-close to $W^u(\omega(N_L)\times(1-\rho,1+\rho))$ in the vicinity of Γ^1. Note that $\dim W^u(\omega(N_L)\times(1-\rho,1+\rho)) = \dim M_L(\varepsilon,\delta) = 4$.

It remains to show that $W^u(\omega(N_L)\times(1-\rho,1+\rho))$ intersects $M_R(\varepsilon)$ transversally since $M_L(\varepsilon,\delta)$ is C^1 $O(\varepsilon)$-close to $W^u(\omega(N_L)\times(1-\rho,1+\rho))$. Recall that, for $\varepsilon = 0$, M_R intersects $W^u(\mathcal{Z})$ transversally along N_R (Proposition 3.2); in particular, at $\gamma_1 := \alpha(\Gamma^1) \in \alpha(N_R) \subset \mathcal{Z}$, we have

$$T_{\gamma_1} M_R = T_{\gamma_1}\alpha(N_R) + T_{\gamma_1} W^u(\gamma_1) + \mathrm{span}\{V_s\}$$

where, $T_{\gamma_1} W^u(\gamma_1)$ is the tangent space of the one-dimensional unstable fiber $W^u(\gamma_1)$ at γ_1 and the vector $V_s \notin T_{\gamma_1} W^u(\mathcal{Z})$ (the latter follows from the transversality of the intersection of M_R and $W^u(\mathcal{Z})$). Also,

$$T_{\gamma_1} W^u(\omega(N_L)\times(1-\rho,1+\rho)) = T_{\gamma_1}(\omega(N_L)\cdot 1) + \mathrm{span}\{V_\tau\} + T_{\gamma_1} W^u(\gamma_1)$$

where the vector V_τ is the tangent vector to the τ-axis as the result of the interval factor $(1-\rho, 1+\rho)$. Recall from Proposition 3.6 that $\omega(N_L)\cdot 1$ and $\alpha(N_R)$ are transversal on $\mathcal{Z} \cap \{\tau = 1\}$. Therefore, at γ_1, the tangent spaces $T_{\gamma_1} M_R$ and $T_{\gamma_1} W^u(\omega(N_L)\times(1-\rho,1+\rho))$ contain seven linearly independent vectors: V_s, V_τ, $T_{\gamma_1} W^u(\gamma_1)$ and the other four from $T_{\gamma_1}(\omega(N_L)\cdot 1)$ and $T_{\gamma_1}\alpha(N_R)$; that is, M_R and $W^u(\omega(N_L)\times(1-\rho,1+\rho))$ intersect transversally. We thus conclude that, there exists $0 < \varepsilon_0 \leq \varepsilon_1$ so that, if $0 < \varepsilon \leq \varepsilon_0$, then $M_L(\varepsilon,\delta)$ intersects $M_R(\varepsilon)$ transversally.

For uniqueness, note that the transversality of the intersection $M_L(\varepsilon,\delta)\bigcap M_R(\varepsilon)$ implies $\dim(M_L(\varepsilon,\delta)\bigcap M_R(\varepsilon)) = \dim M_L(\varepsilon,\delta) + \dim M_R(\varepsilon) - 7 = 1$. Thus, there exists $\delta_0 > 0$ such that, if $0 < \delta \leq \delta_0$, the intersection $M_L(\varepsilon,\delta)\bigcap M_R(\varepsilon)$ consists of precisely one solution near the singular orbit $\Gamma^0 \cup \Lambda \cup \Gamma^1$. □

4. **Ion size effects on individual fluxes and on the current.** The analysis in the previous sections not only establishes the existence of solutions for BVP (2.8)-(2.9) but also provides sufficiently quantitative information on the solution that allows us to extract useful approximations to the *individual* fluxes $\mathcal{J}_i$'s and the *total* flux of charge (current) I for small ν.

In this section, we will study ion size effects on the individual fluxes $\mathcal{J}_i$'s as well as the I-V relations. Contributions to I_1 from $\mathcal{J}_{k1}$ are carefully examined, which provide detailed information on how different ion species interact within ion channels.

We express the flux $\mathcal{J}_i$ as

$$\mathcal{J}_i(V;\lambda,\varepsilon,\nu) = \mathcal{J}_{i0}(V;\varepsilon) + \mathcal{J}_{i1}(V;\lambda,\varepsilon)\nu + o(\nu), \tag{4.1}$$

and the I-V relations defined in (2.11) as

$$\mathcal{I}(V;\lambda,\varepsilon,\nu)=\mathcal{I}_0(V;\varepsilon)+\mathcal{I}_1(V;\lambda,\varepsilon)\nu+o(\nu). \tag{4.2}$$

The term $\mathcal{I}_0(V;\varepsilon)$ (resp. $\mathcal{J}_{i0}$) is the I-V relations (resp. the individual flux) without counting the ion size effect, and $\mathcal{I}_1(V;\lambda,\varepsilon)$ (resp. $\mathcal{J}_{i1}$) is the main term providing effects on the I-V relation (resp. the individual flux) from ion sizes.

We comment that, in (4.1) and (4.2), $\mathcal{J}_i$, $\mathcal{J}_{i0}$, $\mathcal{J}_{i1}$, $\mathcal{I}$, $\mathcal{I}_0$ and $\mathcal{I}_1$ all depend on L_1, L_2, R_1 and R_2 too.

4.1. **Ion size effects on individual fluxes $\mathcal{J}_i$'s.** We introduce two functions $\mathcal{F}_1=\mathcal{F}_1(L_1,L_2;R_1,R_2;\lambda)$ and $\mathcal{F}_2=\mathcal{F}_2(L_1,L_2;R_1,R_2;\lambda)$ as

$$\begin{aligned}\mathcal{F}_1=&\frac{1}{H(1)(\ln c_{10}^L-\ln c_{10}^R)}\left(H(1)\mathcal{F}_2+\frac{z_1(c_{10}^L-c_{10}^R)(R_1-L_1+\lambda(R_2-L_2))}{\ln c_{10}^L-\ln c_{10}^R}\right),\\ \mathcal{F}_2=&\frac{c_{10}^L(L_1+\lambda L_2)-c_{10}^R(R_1+\lambda R_2)}{H(1)}+\frac{z_1\lambda-z_2}{2z_2H(1)}(c_{10}^L-c_{10}^R)(c_{10}^L+c_{10}^R),\end{aligned}$$

where $H(1)=\int_0^1 h^{-1}(s)ds$ in (3.16). We have

Corollary 4.1. *In* (4.1),

$$\begin{aligned}\mathcal{J}_{10}(V;0)=&\frac{D_1(c_{10}^L-c_{10}^R)}{H(1)}+\frac{z_1D_1(c_{10}^L-c_{10}^R)(\ln(L_1R_2)-\ln(L_2R_1))}{(z_1-z_2)H(1)(\ln c_{10}^L-\ln c_{10}^R)}\\ &+\frac{z_1D_1(c_{10}^L-c_{10}^R)}{H(1)(\ln c_{10}^L-\ln c_{10}^R)}\frac{e}{k_BT}V,\\ \mathcal{J}_{20}(V;0)=&-\frac{z_1D_2(c_{10}^L-c_{10}^R)}{z_2H(1)}-\frac{z_1D_2(c_{10}^L-c_{10}^R)(\ln(L_1R_2)-\ln(L_2R_1))}{(z_1-z_2)H(1)(\ln c_{10}^L-\ln c_{10}^R)}\\ &-\frac{z_1D_2(c_{10}^L-c_{10}^R)}{H(1)(\ln c_{10}^L-\ln c_{10}^R)}\frac{e}{k_BT}V,\\ \mathcal{J}_{11}(V;0)=&D_1\left(\mathcal{F}_1\frac{e}{k_BT}V+\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2}\mathcal{F}_1+\mathcal{F}_2\right),\\ \mathcal{J}_{21}(V;0)=&D_2\left(-\mathcal{F}_1\frac{e}{k_BT}V-\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2}\mathcal{F}_1-\frac{z_1}{z_2}\mathcal{F}_2\right),\end{aligned}$$

where c_{10}^L and c_{10}^R are given in Proposition 3.2.

Under electroneutrality boundary conditions (3.9), one has,

$$\begin{aligned}\mathcal{J}_{10}=&\frac{D_1(L-R)\left(z_1\frac{e}{k_BT}V+\ln L-\ln R\right)}{z_1H(1)(\ln L-\ln R)},\quad \mathcal{J}_{20}=\frac{D_2(L-R)\left(z_2\frac{e}{k_BT}V+\ln L-\ln R\right)}{z_2H(1)(\ln L-\ln R)},\\ \mathcal{J}_{11}=&\frac{D_1(z_1\lambda-z_2)}{z_1z_2H(1)}\gamma_0(L,R)\gamma_1(L,R)\frac{e}{k_BT}V-\frac{D_1(z_1\lambda-z_2)(L^2-R^2)}{2z_1^2z_2H(1)},\\ \mathcal{J}_{21}=&-\frac{D_2(z_1\lambda-z_2)}{z_1z_2H(1)}\gamma_0(L,R)\gamma_1(L,R)\frac{e}{k_BT}V+\frac{D_2(z_1\lambda-z_2)(L^2-R^2)}{2z_1z_2^2H(1)},\end{aligned}$$

where

$$\gamma_0(L,R)=\frac{L-R}{\ln L-\ln R},\quad \gamma_1(L,R)=\frac{L-R}{\ln L-\ln R}-\frac{L+R}{2}. \tag{4.3}$$

Proof. From $\mathcal{J}_k=D_kJ_k=D_kJ_{k0}+D_kJ_{k1}\nu+o(\nu)$, one has

$$\mathcal{J}_{k0}(V;0)=D_kJ_{k0}(V;0)\quad\text{and}\quad \mathcal{J}_{k1}(V;0)=D_kJ_{k1}(V;0).$$

The formulas then follow directly from Lemmas 3.4 and 3.5. The conclusion under electroneutrality boundary conditions (3.9) then follows from Proposition 3.2 and Corollary 4.1. □

Remark 4.2. We stress that the linear dependence of $\mathcal{J}_{i0}$'s and $\mathcal{J}_{i1}$'s on V in Corollary 4.1 is due to the fact that they are zeroth order approximations in ε of the corresponding quantities with zero permanent charge. In general, they are nonlinear when permanent charge is non-zero (see, e.g, [43]) and, even with zero permanent charge, higher order terms are not linear in V (see, e.g., [1, 87]).

For the functions defined in (4.3), we have

Lemma 4.3. *If $L \neq R$, then $\gamma_0(L,R) > 0$ and $\gamma_1(L,R) < 0$. As $|L-R| \to 0$ with R being fixed,*

$$\gamma_0(L,R) \to R \text{ and } \gamma_1(L,R) \to 0.$$

Proof. The proof is straightforward. □

Based on the approximations for $\mathcal{J}_i$'s in Corollary 4.1, we define *four* critical potentials and discuss their roles in characterizing ion size effects on individual fluxes.

Definition 4.4. We define four potentials V_{1c}, V_{2c}, V^{1c} and V^{2c} by

$$\mathcal{J}_{11}(V_{1c};\lambda,0) = \mathcal{J}_{21}(V_{2c};\lambda,0) = \frac{d}{d\lambda}\mathcal{J}_{11}(V^{1c};\lambda,0) = \frac{d}{d\lambda}\mathcal{J}_{21}(V^{2c};\lambda,0) = 0.$$

Corollary 4.5. *Suppose $c_{10}^L \neq c_{10}^R$. One has*

$$\begin{aligned}
V_{1c} =& -\frac{k_BT}{e}\left(\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2} + \frac{\mathcal{F}_2}{\mathcal{F}_1}\right),\\
V_{2c} =& -\frac{k_BT}{e}\left(\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2} + \frac{z_1\mathcal{F}_2}{z_2\mathcal{F}_1}\right),\\
V^{1c} =& -\frac{k_BT}{e}\left(\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2} + \frac{(\ln c_{10}^L - \ln c_{10}^R)\mathcal{G}_0}{z_1\left(\mathcal{G}_0 + (R_2-L_2)(c_{10}^L - c_{10}^R)\right)}\right),\\
V^{2c} =& -\frac{k_BT}{e}\left(\frac{\ln(L_1R_2)-\ln(L_2R_1)}{z_1-z_2} + \frac{(\ln c_{10}^L - \ln c_{10}^R)\mathcal{G}_0}{z_2\left(\mathcal{G}_0 + (R_2-L_2)(c_{10}^L - c_{10}^R)\right)}\right),
\end{aligned}$$

where

$$\mathcal{G}_0 = (\ln c_{10}^L - \ln c_{10}^R)\left(L_2c_{10}^L - R_2c_{10}^R + \frac{z_1}{2z_2}(c_{10}^L - c_{10}^R)(c_{10}^L + c_{10}^R)\right).$$

Under the electroneutrality boundary conditions (3.9) and $L \neq R$, one has,

$$V_{1c} = V^{1c} = \frac{k_BT}{e}\frac{L^2-R^2}{2z_1\gamma_0(L,R)\gamma_1(L,R)}, \quad V_{2c} = V^{2c} = \frac{k_BT}{e}\frac{L^2-R^2}{2z_2\gamma_0(L,R)\gamma_1(L,R)}.$$

Proof. The statements follows from Corollary 4.1 and Definition 4.4. □

Remark 4.6. It follows from Corollary 4.5 that $V_{kc} \neq V^{kc}$ and V_{kc} depends on λ in general, but, *under electroneutrality boundary conditions,* $V_{kc} = V^{kc}$ and V_{kc} is independent of λ.

The significance of the four critical values V_{1c}, V_{2c}, V^{1c} and V^{2c} is apparent from their definitions. The value V_{1c} and V_{2c} are the potentials that balance the ion size effects on individual fluxes, and the values V_s^{1c} and V_s^{2c} are the potentials that separate the relative size effects on individual fluxes. More precisely,

Theorem 4.7. *Suppose* $\frac{\partial \mathcal{J}_{k1}}{\partial V}(V;\lambda,0) > 0$ $\big($*resp.* $\frac{\partial \mathcal{J}_{k1}}{\partial V}(V;\lambda,0) < 0\big)$. *For small* $\varepsilon > 0$ *and* $\nu > 0$, *one has*

(i) *if* $V > V_{kc}$ (*resp.* $V < V_{kc}$), *then* $\mathcal{J}_{k1}(V;\varepsilon,\nu) > \mathcal{J}_{k1}(V;\varepsilon,0)$ (*resp.* $\mathcal{J}_{k1}(V;\varepsilon,\nu) < \mathcal{J}_{k1}(V;\varepsilon,0)$);

(ii) *if* $V < V_{kc}$ (*resp.* $V > V_{kc}$), *then* $\mathcal{J}_{k1}(V;\varepsilon,\nu) < \mathcal{J}_{k1}(V;\varepsilon,0)$ (*resp.* $\mathcal{J}_{k1}(V;\varepsilon,\nu) > \mathcal{J}_{k1}(V;\varepsilon,0)$).

Theorem 4.8. *Suppose* $\frac{\partial^2(\mathcal{J}_{k1})}{\partial V\partial\lambda}(V;\lambda,0) > 0$ $\Big($*resp.* $\frac{\partial^2(\mathcal{J}_{k1})}{\partial V\partial\lambda}(V;\lambda,0) < 0\Big)$. *For small* $\varepsilon > 0$ *and* $\nu > 0$, *one has*

(i) *if* $V > V^{kc}$ (*resp.* $V < V^{kc}$), *then* $\mathcal{J}_{k1}$ *is increasing (resp. decreasing)* λ;

(ii) *if* $V < V^{kc}$ (*resp.* $V > V^{kc}$), *then* $\mathcal{J}_{k1}$ *is decreasing (resp. increasing)* λ.

Concerning the conditions in Theorems 4.7 and 4.8, the following result can be easily checked.

Lemma 4.9. *Assume electroneutrality boundary conditions* (3.9) *with* $L \neq R$. *One has, for* $k = 1,2$, $\partial_V\mathcal{J}_{k1} > 0$ *and* $\partial^2_{V\lambda}\mathcal{J}_{k1} > 0$. *As* $L \to R$, $\partial_V\mathcal{J}_{k1} \to 0$ *and* $\partial^2_{V\lambda}\mathcal{J}_{k1} = O((L-R)^2)$. □

4.2. **Ion size effects on the current $\mathcal{I}$.** We analyze ion size effects on the I-V relations following the outline as that in [51].

Corollary 4.10. *In formulas* (4.2), *one has*

$$\begin{aligned}
\mathcal{I}_0(V;0) =& \frac{z_1(z_1D_1 - z_2D_2)(c_{10}^L - c_{10}^R)(\ln(L_1R_2) - \ln(L_2R_1))}{H(1)(\ln c_{10}^L - \ln c_{10}^R)} \\
&+ \frac{z_1(D_1 - D_2)(c_{10}^L - c_{10}^R)}{H(1)} + \left(\frac{z_1(z_1D_1 - z_2D_2)(c_{10}^L - c_{10}^R)}{H(1)(\ln c_{10}^L - \ln c_{10}^R)}\right)\frac{e}{k_BT}V, \\
\mathcal{I}_1(V;0) =& \frac{z_1(z_1\lambda - z_2)(D_1 - D_2)}{z_2H(1)}\left(c_{10}^RR_1 - c_{10}^LL_1 + \frac{1}{2}(c_{10}^L + c_{10}^R)(c_{10}^L - c_{10}^R)\right) \\
&+ \frac{z_1(z_1\lambda - z_2)(z_1D_1 - z_2D_2)(c_{10}^L - c_{10}^R)}{z_2H(1)(\ln c_{10}^L - \ln c_{10}^R)}\left(\frac{c_{10}^RR_1 - c_{10}^LL_1}{c_{10}^L - c_{10}^R} + \frac{c_{10}^L + c_{10}^R}{2}\right. \\
&\left.+ \frac{L_1 - R_1}{\ln c_{10}^L - \ln c_{10}^R}\right)\frac{e}{k_BT}V,
\end{aligned}$$

where c_{10}^L *and* c_{10}^R *are given in Proposition 3.2.*

Under electroneutrality boundary conditions (3.9), *one has*

$$\begin{aligned}
\mathcal{I}_0(V;0) =& \frac{(D_1 - D_2)(L - R)}{H(1)} + \frac{z_1D_1 - z_2D_2}{H(1)}\gamma_0(L,R)\frac{e}{k_BT}V, \\
\mathcal{I}_1(V;\lambda,0) =& -\frac{(z_1\lambda - z_2)(D_1 - D_2)(L^2 - R^2)}{2z_1z_2H(1)} \\
&+ \frac{(z_1\lambda - z_2)(z_1D_1 - z_2D_2)}{z_1z_2H(1)}\gamma_0(L,R)\gamma_1(L,R)\frac{e}{k_BT}V.
\end{aligned}$$

In particular, as $L \to R$, one has

$$\mathcal{I}_0(V;0) \to \frac{(z_1D_1 - z_2D_2)L}{H(1)}\frac{e}{k_BT}V \quad \text{and} \quad \mathcal{I}_1(V;\lambda,0) \to 0.$$

Proof. It follows from

$$\begin{aligned}\mathcal{I}(V;\lambda,0,\nu) =& z_1\mathcal{J}_1 + z_2\mathcal{J}_2 = z_1D_1J_1 + z_2D_2J_2 \\ =& (z_1D_1J_{10} + z_2D_2J_{20}) + (z_1D_1J_{11} + z_2D_2J_{21})\,\nu + o(\nu)\end{aligned} \tag{4.4}$$

that

$$\mathcal{I}_0(V;0) = z_1\mathcal{J}_{10} + z_2\mathcal{J}_{20}, \quad \mathcal{I}_1(V;\lambda,0) = z_1\mathcal{J}_{11} + z_2\mathcal{J}_{21}.$$

The formulas for $\mathcal{I}_0(V;0)$ and $\mathcal{I}_1(V;0)$ follow directly from Lemmas 3.4 and 3.5. □

Similar to Remark 4.2, the zeroth order (in ε) approximation of I-V relation in Corollary 4.10 under the setup of this paper is linear in V. In general, the I-V relation is not linear in V.

We next define three critical potentials V_0, V_c and V^c, which play an important role in characterizing the effect on the I-V relation from ion sizes.

Definition 4.11. We define three potentials V_0, V_c and V^c by

$$\mathcal{I}_0(V_0;0) = 0, \quad \mathcal{I}_1(V_c;\lambda,0) = 0, \quad \frac{d}{d\lambda}\mathcal{I}_1(V^c;\lambda,0) = 0.$$

From Definition 4.11, we obtain

Proposition 4.12. *The potentials V_0, V_c and V^c have the following expressions*

$$\begin{aligned}V_0 =& -\frac{k_BT}{e}\left(\frac{D_1 - D_2}{z_1D_1 - z_2D_2}(\ln c_{10}^L - \ln c_{10}^R) + \ln(L_1R_2) - \ln(L_2R_1)\right), \\ V_c =& V^c = -\frac{k_BT}{e}\frac{(D_1 - D_2)(\ln c_{10}^L - \ln c_{10}^R)\left(\frac{c_{10}^RR_1 - c_{10}^LL_1}{c_{10}^L - c_{10}^R} + \frac{c_{10}^L + c_{10}^R}{2}\right)}{(z_1D_1 - z_2D_2)\left(\frac{c_{10}^RR_1 - c_{10}^LL_1}{c_{10}^L - c_{10}^R} + \frac{c_{10}^L + c_{10}^R}{2} + \frac{L_1 - R_1}{\ln c_{10}^L - \ln c_{10}^R}\right)}.\end{aligned}$$

Under electroneutrality conditions (3.9) and $L \neq R$, one has

$$\begin{aligned}V_0 =& -\frac{k_BT}{e}\frac{D_1 - D_2}{z_1D_1 - z_2D_2}(\ln L - \ln R), \\ V_c =& \frac{k_BT}{e}\frac{(D_1 - D_2)(L^2 - R^2)}{2(z_1D_1 - z_2D_2)\gamma_0(L,R)\gamma_1(L,R)}.\end{aligned}$$

For the LHS used in [51], $V_c \neq V^c$ in general. In the following, we will use the notion V_c for Bikerman's LHS taken in this paper.

As a direct consequence of Proposition 4.12, one has

Corollary 4.13. *Assume electroneutrality boundary conditions (3.9). Then*

(i) *$V_0(L,R) = -V_0(R,L)$ and $V_c(L,R;\lambda) = -V_c(R,L;\lambda)$;*

(ii) *for $L \geq R$, $V_0(L,R)$ is decreasing (resp. increasing) in L if $D_1 > D_2$ (resp. $D_1 < D_2$), and, for fixed $R > 0$, $\lim_{L\to R} V_0(L,R) = 0$;*

(iii) *for fixed $R > 0$,*

$$\begin{aligned}&\lim_{L\to R} V_c(\ln L - \ln R) = -\frac{12k_BT}{e}\frac{D_1 - D_2}{z_1D_1 - z_2D_2}; \\ &\lim_{L\to\infty}\frac{V_c}{\ln L - \ln R} = -\frac{k_BT}{e}\frac{D_1 - D_2}{z_1D_1 - z_2D_2}.\end{aligned}$$

A direct observation gives the following result:

Theorem 4.14. *Treating $\mathcal{I}_0$, $\mathcal{I}_1, V_0$, and V_c as functions of (L_1, L_2, R_1, R_2), one has*

(i) *$\mathcal{I}_0$ is homogeneous of degree one, that is, for any $s > 0$,*

$$\mathcal{I}_0(V, sL_1, sL_2, sR_1, sR_2; 0) = s\mathcal{I}_0(V, L_1, L_2, R_1, R_2; 0).$$

(ii) *$\mathcal{I}_1$ is homogeneous of degree two, that is, for any $s > 0$,*

$$\mathcal{I}_1(V, sL_1, sL_2, sR_1, sR_2; 0) = s^2\mathcal{I}_1(V, L_1, L_2, R_1, R_2; 0).$$

(iii) *The potentials V_0 and V_c are homogeneous of degree zero.*

The potential V_0 is the so-called *reversal* potential. The value V_c is the potential that balances ion size effect on I-V relations and the value V^c is the potential that separates the relative size effect on I-V relations. Precise statements are provided as follows:

Theorem 4.15. *Suppose $\partial_V\mathcal{I}_1(V; \lambda, 0) > 0$ (resp. $\partial_V\mathcal{I}_1(V; \lambda, 0) < 0$). For small $\varepsilon > 0$ and $\nu > 0$,*

(i) *If $V > V_c$ (resp. $V < V_c$), then $\mathcal{I}(V; \varepsilon, \nu) > \mathcal{I}(V; \varepsilon, 0)$;*
(ii) *If $V < V_c$ (resp. $V > V_c$), then $\mathcal{I}(V; \varepsilon, \nu) < \mathcal{I}(V; \varepsilon, 0)$.*

Similarly,

Theorem 4.16. *Suppose $\partial^2_{V\lambda}\mathcal{I}_1(V; \lambda, 0) > 0$ (resp. $\partial^2_{V\lambda}\mathcal{I}_1(V; \lambda, 0) < 0$). For small $\varepsilon > 0$ and $\nu > 0$,*

(i) *If $V > V^c$ (resp. $V < V^c$), then the current $\mathcal{I}$ is increasing λ;*
(ii) *If $V < V^c$ (resp. $V > V^c$), then the current $\mathcal{I}$ is decreasing λ.*

The following result can be checked easily.

Proposition 4.17. *Assume electroneutrality boundary conditions (3.9) with $L \neq R$. Then, $\partial_V\mathcal{I}_1(V; \lambda, 0) > 0$. As $L \to R$, $\partial_V\mathcal{I}_1(V; \lambda, 0) \to 0$.* □

While $\partial_V\mathcal{I}_1(V; \lambda, 0)$ is non-negative under electroneutrality conditions, in general, it can be negative, as shown in the following example motivated by that in [51]. We consider a special case with $z_1 = 1$ and $z_2 = -1$. Correspondingly, we have $c^L_{10} = \sqrt{L_1L_2}$ and $c^R_{10} = \sqrt{R_1R_2}$.

Proposition 4.18. *Fix $L_2 > 0$. If either $R_2 \geq R_1 \geq L_1 > 0$ and $\sqrt{L_1L_2} > \sqrt{R_1R_2}$, or $R_1 \geq L_1$, $R_2 < R_1$ and $\sqrt{L_1L_2} > \mu^*\sqrt{R_1R_2}$, where $\mu^* > 1$ is a constant, then*

$$\partial_V\mathcal{I}_1(V; \lambda, 0) = -\frac{e}{k_BT}\frac{(\lambda+1)(D_1+D_2)(\sqrt{L_1L_2}-\sqrt{R_1R_2})}{H(1)(\ln(L_1L_2)-\ln(R_1R_2))}\mathcal{K}(L_1, L_2, R_1, R_2)$$

is negative, where

$$\begin{aligned}\mathcal{K}(L_1, L_2, R_1, R_2) =& \frac{L_1 - R_1}{\ln(L_1L_2)-\ln(R_1R_2)} - \frac{L_1\sqrt{L_1L_2}-R_1\sqrt{R_1R_2}}{\sqrt{L_1L_2}-\sqrt{R_1R_2}} \\ &+ \frac{\sqrt{L_1L_2}+\sqrt{R_1R_2}}{2}.\end{aligned} \tag{4.5}$$

Proof. Note that

$$\frac{e(D_1+D_2)}{k_BTH(1)} > 0,\ \lambda > 0,\ \text{and}\ \frac{\sqrt{L_1L_2}-\sqrt{R_1R_2}}{\ln(L_1L_2)-\ln(R_1R_2)} > 0,\ \text{for}\ L_1L_2 \neq R_1R_2,$$

it is suffices to show that $\mathcal{K}(L_1, L_2, R_1, R_2) > 0$.

For simplicity, we set

$$\sqrt{L_1L_2} = \mu\sqrt{R_1R_2}, \tag{4.6}$$

where $\mu > 0$, but $\mu \neq 1$. Substituting (4.6) into equation (4.5), we have

$$\mathcal{K}(L_1, L_2, R_1, R_2) = \frac{h(\mu)}{2(\mu-1)\ln\mu}, \tag{4.7}$$

where

$$h(\mu) = (L_1 - R_1)(\mu - 1) + 2(R_1 - L_1\mu)\ln\mu - \sqrt{R_1R_2}(1-\mu^2)\ln\mu.$$

Notice that $2(\mu-1)\ln\mu > 0,$ for $\mu > 0,$ but $\mu \neq 1$. Now we claim that $h(\mu) > 0$.

To get started, a simple calculation gives

$$h'(\mu) = R_1\left(\frac{2}{\mu} - 1\right) - L_1(1 + 2\ln\mu) + \sqrt{R_1R_2}\left(2\mu\ln\mu + \mu - \frac{1}{\mu}\right).$$

Case I: $R_2 \geq R_1 \geq L_1 > 0,$ **and** $\mu > 1$.

Under the assumption of case 1, one has $h'(\mu) > h_0(\mu)R_1$, where

$$h_0(\mu) = \frac{1}{\mu} - 2 - 2\ln\mu + 2\mu\ln\mu + \mu.$$

A careful calculation gives

$$h_0(1) = h_0'(1) = 0, \text{ and } h_0''(\mu) = \frac{2}{\mu}\left(1 + \frac{1}{\mu} + \frac{1}{\mu^2}\right) > 0, \text{ for } \mu > 1.$$

And hence, $h'(\mu) > 0$ for $\mu > 1$. Together with $h(1) = 0$, we have $h(\mu) > 0$ for $\mu > 1$. Therefore, $\mathcal{K}(L_1, L_2, R_1, R_2) > 0$.

Case II: $R_1 \geq L_1,\ R_2 < R_1,$ **and** $\mu > \mu^* > \mu_0 > 1$.

For convenience, we define $R_2 = aR_1$, where $0 < a < 1$. Then, we have

$$h'(\mu) > g(\mu)R_1 \quad \text{with} \quad g(\mu) = \frac{2-\sqrt{a}}{\mu} + a\mu + 2\sqrt{a}\mu\ln\mu - 2\ln\mu - 2.$$

Direct calculations give

$$g'(\mu) = \frac{\sqrt{a}-2}{\mu^2} + 2\sqrt{a}\ln\mu - \frac{2}{\mu} + 3\sqrt{a} \quad \text{and} \quad g''(\mu) = \frac{2-\sqrt{a}}{2\mu^3} + \frac{2\sqrt{a}}{\mu} + \frac{2}{\mu^2}.$$

Clearly, one has $g''(\mu) > 0$, for all $\mu > 1$. And hence, $g'(\mu)$ is increasing for $\mu > 1$. Note that $g'(1) < 0$, and $g'(\mu) \to \infty$ as $\mu \to \infty$. There exists a unique $\mu_0 > 0$ such that $g'(\mu_0) = 0$. Furthermore, $g(\mu)$ is decreasing for $1 < \mu < \mu_0$ and increasing for $\mu > \mu_0$. Note that $g(1) = 0$, we have $g(\mu_0) < 0$, and there exists a unique $\mu^* > \mu_0$ such that $g(\mu^*) = 0$, and $g(\mu) > 0$ for $\mu > \mu^*$. This completes the proof. □

4.3. **Individual fluxes vs the current.** The critical potential V_c is directly related to V_{kc} and $V^{kc},\ k = 1, 2$. For simplicity, from now on, we always assume the electroneutrality boundary conditions (3.9).

Recall that, under electroneutrality boundary conditions, $V_{kc} = V^{kc}$. We thus use V_{kc} in the following. The next result follows from Corollary 4.5 and Proposition 4.12.

Lemma 4.19. *Assume electroneutrality conditions* (3.9)*. Then*

$$V_c = \frac{z_1D_1V_{1c} - z_2D_2V_{2c}}{z_1D_1 - z_2D_2}.$$

For fixed D_1, D_2, L, and R, one can immediately characterize ion size effects on the individual fluxes $\mathcal{J}_k$'s and the current $\mathcal{I}$, depending on the relative locations of V_c, V_{1c}, V_{2c} and where the boundary potential V is located. We will provide the result only for the cases $D_1 > D_2$. The statements for other cases can be made similarly.

Theorem 4.20. *Assume electroneutrality conditions* (3.9)*. Suppose* $\lambda \neq 1$*, and* $D_1 > D_2$ *and* $L < R$*. Then,*

$$V_c > V_{1c} > V_{2c}.$$

Hence, for small $\varepsilon > 0$ *and* $\nu > 0$*,*

(i) *if* $V > V_c$*, then ion sizes enhance* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,\nu) > \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,\nu) > \mathcal{J}_2(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,\nu) > \mathcal{I}(V;\varepsilon,0);$$

(ii) *if* $V_{1c} < V < V_c$*, then ion sizes enhance* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *but reduce* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,\nu) > \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,\nu) > \mathcal{J}_2(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,\nu) < \mathcal{I}(V;\varepsilon,0);$$

(iii) *if* $V_{2c} < V < V_{1c}$*, then ion sizes enhance* $\mathcal{J}_2$ *but reduce* $\mathcal{J}_1$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,\nu) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,\nu) > \mathcal{J}_2(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,\nu) < \mathcal{I}(V;\varepsilon,0);$$

(iv) *if* $V < V_{2c}$*, then ion sizes reduce* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,\nu) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,\nu) < \mathcal{J}_2(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,\nu) < \mathcal{I}(V;\varepsilon,0).$$

Proof. The relation among V_c, V_{1c}, and V_{2c} follows from Corollary 4.5, Proposition 4.12, Lemma 4.19, and the assumption that $D_1 > D_2$ and $L < R$.

The statements (i)–(iv) then follow from Theorems 4.7 and 4.15. □

Remark 4.21. For cases (i) and (ii) in Theorem 4.20, ion size effects on individual fluxes $\mathcal{J}_1$ and $\mathcal{J}_2$ are the same, but their effects on the current $\mathcal{I}$ are opposite. For cases (iii) and (iv), ion size effects on the flux $\mathcal{J}_2$ are opposite, but their effects on the flux $\mathcal{J}_1$ and the current $\mathcal{I}$ are the same.

Similarly, one has

Theorem 4.22. *Assume electroneutrality conditions* (3.9)*. Suppose* $\lambda \neq 1$*, and* $D_1 > D_2$ *and* $L > R$*. Then,*

$$V_c < V_{1c} < V_{2c}.$$

Hence, for small $\varepsilon > 0$ *and* $\nu > 0$*,*

(i) *if* $V < V_c$*, then ion sizes reduce* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,d) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,d) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,d) < \mathcal{I}(V;\varepsilon,0).$$

(ii) *if* $V_c < V < V_{1c}$*, then ion sizes reduce* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *but enhance* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,d) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,d) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,d) > \mathcal{I}(V;\varepsilon,0).$$

(iii) *if* $V_{1c} < V < V_{2c}$*, then ion sizes reduce* $\mathcal{J}_2$ *but enhance* $\mathcal{J}_1$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,d) > \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,d) < \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,d) > \mathcal{I}(V;\varepsilon,0).$$

(iv) *if* $V > V_{2c}$*, then ion sizes enhance* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$*, that is,*

$$\mathcal{J}_1(V;\varepsilon,d) > \mathcal{J}_1(V;\varepsilon,0),\ \mathcal{J}_2(V;\varepsilon,d) > \mathcal{J}_2(V;\varepsilon,0),\ \mathcal{I}(V;\varepsilon,d) > \mathcal{I}(V;\varepsilon,0).$$

The effects of relative ion size λ on ionic flows can also be derived directly. Recall that under electroneutrality conditions (3.9), we have $V_{ic} = V^{ic}$ for $i = 1, 2$.

Theorem 4.23. *Assume electroneutrality conditions* (3.9)*. Suppose* $D_1 > D_2$ *and* $L < R$*. Then,*

$$V_c > V_{1c} > V_{2c}.$$

Hence, for small $\varepsilon > 0$ *and* $\nu > 0$*, one has*

(i) *if* $V > V_c$*, then* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$ *increase in* λ*;*
(ii) *if* $V_{1c} < V < V_c$*, then* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *increase in* λ *but* $\mathcal{I}$ *decreases in* λ*;*
(iii) *if* $V_{2c} < V < V_{1c}$*, then* $\mathcal{J}_1$ *and* $\mathcal{I}$ *decrease in* λ *but* $\mathcal{J}_2$ *increases in* λ*;*
(iv) *if* $V < V_{2c}$*, then* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$ *decrease in* λ*.*

Similarly,

Theorem 4.24. *Assume electroneutrality conditions* (3.9)*. Suppose* $D_1 > D_2$ *and* $L > R$*. Then,*

$$V_c < V_{1c} < V_{2c}.$$

Hence, for small $\varepsilon > 0$ *and* $\nu > 0$*, one has*

(i) *if* $V < V_c$*, then* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$ *decrease in* λ*;*
(ii) *if* $V_c < V < V_{1c}$*, then* $z_1\mathcal{J}_1$ *and* $\mathcal{J}_2$ *decrease in* λ *but* $\mathcal{I}$ *increases in* λ*;*
(iii) *if* $V_{1c} < V < V_{2c}$*, then* $\mathcal{J}_1$ *and* $\mathcal{I}$ *increase in* λ *but* $\mathcal{J}_2$ *decreases in* λ*;*
(iv) *if* $V > V_{2c}$*, then* $\mathcal{J}_1$*,* $\mathcal{J}_2$ *and* $\mathcal{I}$ *increase in* λ*.*

4.4. **Sensitivity of ion size effects near $L = R$.** We examine closely the situation for L and R close to each other. It turns out, in this situation, the properties of the critical potentials are extremely sensitive on whether $L > R$ or $L < R$.

Proposition 4.25. *One has,*

$$\lim_{L\to R^+} V_{1c} = \lim_{L\to R^-} V_{2c} = -\infty, \quad \lim_{L\to R^-} V_{1c} = \lim_{L\to R^+} V_{2c} = +\infty.$$

Proof. From Lemma 4.3, one has

$$\lim_{L\to R^+} \frac{L^2 - R^2}{\gamma_0(L,R)\gamma_1(L,R)} = -\infty \quad \text{and} \quad \lim_{L\to R^-} \frac{L^2 - R^2}{\gamma_0(L,R)\gamma_1(L,R)} = \infty.$$

Recall that $z_1 > 0 > z_2$. Our results then follow directly from Corollary 4.5. □

The significance of the above result is discussed in the next remark.

Remark 4.26. Combining this result with Theorems 4.20 and Theorems 4.22, one concludes that the effects of ion sizes are sensitive to whether $L > R$ or $L < R$ for L and R close. More precisely, on one hand, as $L \to R^-$, one has $V_{2c} < V < V_{1c}$ for any fixed potential V, and hence, ion sizes always *reduce* $\mathcal{J}_1$ (comparing to $\mathcal{J}_1$ from point-charge case) but *enhance* $\mathcal{J}_2$ (see, (iii) in Theorem 4.20); on the other hand, as $L \to R^+$, exactly the opposite occurs, that is, one has $V_{2c} > V > V_{1c}$ for any fixed potential V, and hence, ion sizes always *enhance* $\mathcal{J}_1$ but *reduce* $\mathcal{J}_2$ (see, (iii) in Theorem 4.22).

Similar sensitivity dependence of ion sizes effects on total fluxes near $L = R$ is examined below. The result depends naturally on D_1 and D_2 as well as z_1 and z_2.

Proposition 4.27. *Assume* $D_1 > D_2$*. One has,*

$$\lim_{L\to R^+} V_c = -\infty \quad \text{and} \quad \lim_{L\to R^-} V_c = +\infty.$$

Proof. It follows from

$$\lim_{L\to R^+}\frac{L-R}{\gamma_1(L,R)}=-\infty \quad \text{and} \quad \lim_{L\to R^-}\frac{L-R}{\gamma_1(L,R)}=\infty.$$

□

Remark 4.28. (a) Similar to Remark 4.26, when combining Proposition 4.27 with Theorems 4.15 and 4.16, and Proposition 4.17, one concludes sensitive dependence of ion size effects on the current $\mathcal{I}$ near $L=R$. The precise dependence further involves the quantities D_1 and D_2; for example, if $D_1 > D_2$, on one hand, as $L\to R^+$, one has $V>V_c$ for any fixed potential V, and hence, ion size always *enhances* the current $\mathcal{I}$ comparing to the current from point-charge case (see, (i) in Theorem 4.15) and the current $\mathcal{I}$ is always increasing in λ (see, (ii) in Theorem 4.16); on the other hand, as $L\to R^-$, exactly the opposite effect occurs. For the other cases, the ion size effects as $L\to R^-$ are always opposite to those as $L\to R^+$.

(b) Comparing consequences from results in Proposition 4.25 and in Proposition 4.27, we note that the qualitatively sensitive dependences of ion sizes on individual fluxes $\mathcal{J}_1 = D_1J_1$ and $\mathcal{J}_2 = D_2J_2$ do not depend on D_1 and D_2 but those on the current $\mathcal{I}$ do, simply because $\mathcal{I} = z_1D_1J_1 + z_2D_2J_2$ with $z_1 > 0 > z_2$.

5. **Concluding remarks.** We study a quasi-one-dimensional steady-state Poisson-Nernst-Planck model for ionic flows through a single membrane channel with two ion species, one positively charged and one negatively charged. Bikerman's local hard-sphere model is included to account for ion size effects. Under the framework of geometric singular perturbation theory, together with the specific structures of the PNP system, approximations to the individual fluxes and the I-V relations are extracted, from which the qualitative properties of ionic flows are studied. A detailed characterization of complicated interactions among multiple and physically crucial parameters (such as boundary concentrations and potentials, diffusion coefficients and ion sizes) for ionic flows is provided. Based on relatively simple biological settings, our results have demonstrated extremely rich behaviors of ionic flows and sensitive dependence of flow properties on all these parameters. We believe that this work will be useful for numerical studies and stimulate further analytical studies of ionic flows through membrane channels.

We finally point out that the approximated I-V relation (zeroth order in ε) is linear in V (See Corollary 4.10) under our set-up. However, the zeroth order (in ε) approximation of the I-V relation is *nonlinear* in V when permanent charge is nonzero (see, [43]) and, even with zero permanent charge, higher order terms are nonlinear in V (see [1, 87] for examples).

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E-mail address: jiayusheng1982@tom.com
E-mail address: wliu@math.ku.edu
E-mail address: mzhang@nmt.edu