Poiseuille flow and apparent viscosity of nematic liquid-crystals

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Abstract

In this work, we study the stationary Poiseuille flow of nematic liquid-crystals through a tube under a constant gradient pressure. We first examine the dynamics of steady-state system of Poiseuille flow with strong anchoring boundary conditions. For zero gradient pressure (equilibrium situation), the steady-state system can be converted essentially to a Hamiltonian system but it is not structurally stable in its natural form. Interestingly, by revealing a special structure of the problem, the system can be converted to one with normal hyperbolicity so that the limiting dynamics is actually structurally stable. In the second part, based on the structures of the system established in the first part, we provide an approximation for the so-called *apparent viscosity* η. As discovered by Atkin [*Arch. Ration. Mech. Anal.* **38** (1970), pp. 224-240] and verified experimentally by Fisher and Fredrickson [*Mol. Cryst. Liq. Cryst.* **8** (1969), pp. 267-284], the apparent viscosity η is in fact a function of the ratio between the efflux and the radius of the tube. But the function is unknown, and it has been approximated numerically for a couple of cases. The approximation procedure provided in this paper allows one to obtain the Taylor expansion of the function to any order and the coefficients in the expansion can be expressed as integrals involving *structural functions* and *physical parameters* of the problem only.

1 Introduction

Liquid-crystals are intermediate phases between solid and liquid states. While liquid-crystals may flow like fluids, they also possess features of solid crystals ([3, 7, 9, 14], etc.). For example, *nematic* and *cholesteric* liquid-crystals consist of molecules of rod-like or disc-like. Based on early works of Oseen, Zöcher in 1930s ([18, 26]), and of Frank in 1950s ([11]), a continuum model for liquid crystals was formulated by Ericksen and Leslie in 1960s ([9, 14]). In this classical continuum

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theory of Ericksen and Leslie for liquid crystals, state variables are the *velocity* $\mathbf{u}(x,t)$ and the *director* $\mathbf{n}(x,t)$. The director field $\mathbf{n}(x,t)$ represents the locally preferred alignment direction at location x and at time t of molecules (capturing the crystal aspect of the material). Using Einstein's summation convention, the governing system for nematic liquid crystals is (see [3, 7, 9, 14, 21, 25])

$$\begin{aligned} \rho\dot{u}_i &= F_i + t_{ij,j}, \\ \sigma\ddot{n}_i &= G_i + g_i + s_{ij,j}, \\ u_{j,j} &= 0, \quad n_j n_j = 1, \end{aligned} \tag{1.1}$$

where $\mathbf{F} = (F_i)$ is the (external) body force per unit volume, $\mathbf{G} = (G_i)$ the generalized or director body force per unit volume; $\mathbf{t} = (t_{ij})$ the stress tensor, $\mathbf{s} = (s_{ij})$ a director stress tensor that is intimately related to the couple stress acting on liquid crystal surface, $\mathbf{g} = (g_i)$ the director body force vector.

The constitutive relations for $\mathbf{t}$, $\mathbf{s}$ and $\mathbf{g}$ are

$$\begin{aligned} t_{ij} =& - p\delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + t^D_{ij}, \quad s_{ij} = n_i\beta_j + \frac{\partial W}{\partial n_{i,j}}, \\ g_i =& \gamma n_i - (n_i\beta_j)_{,j} - \frac{\partial W}{\partial n_i} + g^D_i, \quad e_{ijk}(t^D_{kj} - n_j g^D_k) = 0, \end{aligned} \tag{1.2}$$

where $W(\mathbf{n}, \nabla\mathbf{n})$ is the static energy function, $\mathbf{t}^D$ and $\mathbf{g}^D$ the respective dynamical components of $\mathbf{t}$ and $\mathbf{g}$, p the pressure, γ the Lagrange multiplier due to the constraint $|\mathbf{n}| = 1$, and the vector (β_j) an intermediate variable that will not affect the system.

The static energy function $W(\mathbf{n}, \nabla\mathbf{n})$ satisfies, for any orthogonal matrix Q with $Q^{-1} = Q^T$,

$$W(Q\mathbf{n}, Q\nabla\mathbf{n}Q^T) = W(\mathbf{n}, \nabla\mathbf{n}),$$

and is, up to quadratic terms in $\nabla\mathbf{n}$, given by the Frank's formula

$$2W(\mathbf{n}, \nabla\mathbf{n}) = K_1(\operatorname{div}\mathbf{n})^2 + K_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + K_3(\mathbf{n}\cdot\nabla)\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{n}, \tag{1.3}$$

with $K_2 > K_1 > 0$ and $K_3 > 0$; with the assumption that $\mathbf{t}^D$ can be approximated by its linear term, it is given by

$$\begin{aligned} t^D_{ij} =& \alpha_1 n_k A_{kp} n_p n_i n_j + \alpha_2 N_i n_j + \alpha_3 N_j n_i + \alpha_4 A_{ij} \\ &+ \alpha_5 A_{ik} n_k n_j + \alpha_6 n_i A_{jk} n_k, \end{aligned} \tag{1.4}$$

$$g^D_i = - \gamma_1 N_i - \gamma_2 A_{ik} n_k, \quad \gamma_1 = \alpha_3 - \alpha_2,\ \gamma_2 = \alpha_6 - \alpha_5, \tag{1.5}$$

and

$$2A_{ij} = u_{i,j} + u_{j,i}, \quad 2S_{ij} = u_{i,j} - u_{j,i}, \quad N_i = \dot{n}_i - S_{ik}n_k;$$

The parameters α_j's meet the following empirical relations (p.13, [14])

$$\begin{aligned}
&\alpha_4 > 0, \quad 2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 > 0, \quad \gamma_1 = \alpha_3 - \alpha_2 > 0, \\
&2\alpha_4 + \alpha_5 + \alpha_6 > 0, \quad 4\gamma_1(2\alpha_4 + \alpha_5 + \alpha_6) > (\alpha_2 + \alpha_3 + \gamma_2)^2, \\
&\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5 = \gamma_2.
\end{aligned} \tag{1.6}$$

The last relation is called the Onsager-Parodi relation ([19]).

In [2], Atkin applied the continuum theory to study the Poiseuille flow of liquid crystals down a tube driving by a constant pressure gradient. Under some assumptions, Atkin used a scaling argument to show that the *apparent viscosity* η is a function $\eta = \mathcal{S}(Q/R)$ of the ratio Q/R of the volume flux Q and the radius R of the tube. This result reveals a significant difference between (anisotropic) liquid crystal fluids and isotropic fluids: for the latter, Coleman and Noll ([4]) showed that the apparent viscosity is a function of Q/R^3. Atkin's prediction has been tested experimentally by Fisher and Fredrickson ([12]). For the case of perpendicular boundary orientation, remarkably good agreement was found between the theoretical predication and the experimental data. However, there is a discrepancy between the theory and experiment for the parallel orientation. In [5], Currie explored possible explanations of this discrepancy. Based on experimental data for some of the structural functions in the system, Currie considered an approximation of the system for Poiseuille flow that allowed him to obtain explicit information. His analysis suggested that the discrepancy might be due to the experimental inaccuracy rather than a failure of the theory.

In this work, we study the Poiseuille flow of nematic liquid-crystals using modern theory of dynamical systems. As remarked in [2, 5, 14], etc., the governing system for the stationary Poiseuille flow seems not amenable to analytic methods. We will show that this is not the case with two new observations. The first observation is that, for zero gradient pressure (equilibrium situation), the steady-state system can be cast as a decoupled system of a Hamiltonian system and a trivial system. On the other hand, this system is not structurally stable that prevents a direct application of the invariant manifold theory to non-zero gradient pressure cases. The second observation is the non-zero gradient pressure system is a *special* perturbation of the zero gradient pressure system. It turns out one can find a *non-smooth* change of variables so that the new system becomes structurally stable. Luckily, the non-smooth change of variables does not affect the results interested. This is accomplished in §2. Based on the result in §2 and the dynamical system theory, we then provide an approximation for the *apparent viscosity* η in §3. The approximation procedure provided in this work allows one to obtain the Taylor expansion of the function $\eta = \mathcal{S}(Q/R)$ to any degree and the coefficients in the expansion are expressed in terms of *structural functions* and *physical parameters* of the problem only.

2 Stationary Poiseuille flow

Consider the stationary Poiseuille flow of nematic liquid-crystals through a fixed tube of radius R driving by a constant pressure gradient down the tube. We look for solutions of the form, in terms of cylindrical coordinates,

$$\mathbf{u} = (u_r, u_\phi, u_z) = (0, 0, u(r)), \quad \mathbf{n} = (n_r, n_\phi, n_z) = (\sin\theta(r), 0, \cos\theta(r)),$$

where θ is the angle between $\mathbf{n}$ and the positive z-axis. The system for the stationary Poiseuille flow becomes ([2, 14]),

$$\begin{aligned} &g(\theta)u' = -\frac{ar}{2} + \frac{b}{r}, \\ &f(\theta)\theta'' + \frac{f_\theta(\theta)}{2}\theta'^2 + \frac{f(\theta)}{r}\theta' - \frac{K_1\sin(2\theta)}{2r^2} - \frac{\gamma_1 + \gamma_2\cos(2\theta)}{2}u' = 0 \end{aligned} \tag{2.7}$$

where $a \geq 0$ is the magnitude of the constant pressure gradient, b is an integrating constant,

$$\begin{aligned} g(\theta) =&\alpha_1\sin^2\theta\cos^2\theta + \frac{\alpha_5 - \alpha_2}{2}\sin^2\theta + \frac{\alpha_3 + \alpha_6}{2}\cos^2\theta + \frac{\alpha_4}{2}, \\ f(\theta) =&K_1\cos^2\theta + K_3\sin^2\theta, \quad h(\theta) = \alpha_3\cos^2\theta - \alpha_2\sin^2\theta. \end{aligned} \tag{2.8}$$

For Poiseuille flow through two concentric circular tubes of radii R and R_0 with $R > R_0$, Atkin ([2]) established the existence and uniqueness of a solution for some boundary value problems. If $R_0 = 0$, Atkin provided an analysis that is not complete. We will focus on the Poiseuille flow for this situation. If we consider $R_0 = 0$, then $b = 0$ from the first equation in (2.7), and hence, the steady-state system becomes

$$u' = -\frac{ar}{2g(\theta)}, \tag{2.9}$$

$$f(\theta)\theta'' + \frac{f_\theta(\theta)}{2}\theta'^2 + \frac{f(\theta)}{r}\theta' - \frac{K_1\sin(2\theta)}{2r^2} + \frac{a(\gamma_1 + \gamma_2\cos(2\theta))r}{4g(\theta)} = 0. \tag{2.10}$$

The boundary conditions are

$$u(R) = 0, \quad \theta(R) = \theta_0, \tag{2.11}$$

for some $\theta_0 \in [0, \pi)$. Note that system (2.16) is periodic in θ with period π reflecting head-tail symmetry of nematic material. In the sequel, we will take $\theta_0 \in [0, \pi/2]$. The boundary condition $u(R) = 0$ reflects stationary of the tube and $\theta(R) = \theta_0$ is the so-called *strong anchoring* boundary condition.

The Frank's energy function (1.3) in this case is

$$W(\mathbf{n}, \nabla\mathbf{n}) = K_1\left(c^2(\theta)\theta'^2 + \frac{2s(\theta)c(\theta)}{r}\theta' + \frac{s^2(\theta)}{r^2}\right) + K_3 s^2(\theta)\theta'^2.$$

The bulk energy is then given by

$$\tilde{\mathcal{E}}(\theta) = \int_0^R \left(f(\theta)\theta'^2 + \frac{K_1 s^2(\theta)}{r^2} \right) r\, dr + K_1 \int_0^R \left(s^2(\theta) \right)'\, dr.$$

In order to have a finite energy $\tilde{\mathcal{E}}(\theta)$, necessarily, $\theta(r) \to 0$ as $r \to 0^+$. We thus have, using $\theta(R) = \theta_0$,

$$\tilde{\mathcal{E}}(\theta) = \int_0^R \left(f(\theta)\theta'^2 + \frac{K_1 s^2(\theta)}{r^2} \right) r\, dr + K_1 s^2(\theta_0).$$

For *strong anchoring* boundary conditions with the requirement $\theta(r) \to 0$ as $r \to 0^+$, we simply take the bulk energy

$$\mathcal{E}(\theta) = \int_0^R \left(f(\theta)\theta'^2 + \frac{K_1 s^2(\theta)}{r^2} \right) r\, dr. \tag{2.12}$$

Equation (2.10) for θ decoupled from that for u. If $\theta(r; a, R)$ is the solution of (2.10) with $\theta(R; a, R) = \theta_0$ and $\theta(r; a, R) \to 0$ as $r \to 0$, one can determine u by integration; in particular, for $u(R) = 0$, one has

$$u(r) = u(R) - a \int_R^r \frac{s}{2g(\theta(s; a, R))} ds = \frac{a}{2} \int_r^R \frac{s}{g(\theta(s; a, R))} ds. \tag{2.13}$$

2.1 An equivalent boundary value problem

Our first step in studying the boundary value problem is to rewrite (2.10) into a special form of an autonomous system of first order equations. Thus we consider the following equivalent boundary value problem

$$\begin{aligned} \theta' &= \frac{1}{r f(\theta)} \rho, \\ \rho' &= \frac{f_\theta(\theta)}{2r f^2(\theta)} \rho^2 + K_1 \frac{\sin(2\theta)}{2r} - \frac{a(\gamma_1 + \gamma_2 \cos(2\theta)) r^2}{4g(\theta)}, \end{aligned} \tag{2.14}$$

with

$$\theta(R) = \theta_0 \ \text{ and } \ \theta(r) \to 0 \ \text{ as } \ r \to 0^+. \tag{2.15}$$

Next, we use the rescale $r = Re^t$ and introduce a new variable $\tau = r/R$. Problem (2.14) and (2.15) is equivalent to

$$\begin{aligned} \theta'(t) &= \frac{1}{f(\theta)} \rho, \\ \rho'(t) &= \frac{f_\theta}{2f^2(\theta)} \rho^2 + K_1 \frac{\sin(2\theta)}{2} - aR^3 \frac{(\gamma_1 + \gamma_2 \cos(2\theta)) \tau^3}{4g(\theta)}, \\ \tau'(t) &= \tau, \end{aligned} \tag{2.16}$$

and

$$\tau(0) = 1,\ \theta(0) = \theta_0 \ \text{ and } \ \theta(t) \to 0 \ \text{ as } \ t \to -\infty. \tag{2.17}$$

Remark 2.1. *(i) The advantage of this special form is that $\{\tau = 0\}$ is invariant for system (2.16) and, on $\{\tau = 0\}$, the (θ, ρ)-system is a Hamiltonian (see Lemma 2.2 below). This observation has not been made before.*

(ii) It seems that, by considering system (2.16), one looses dependence of θ on r. But it does not. In fact, the phase portrait of system (2.16) is exactly the same as that of system (2.14) away from $r = 0$. Thus, once one has an orbit $(\theta(t), \rho(t), \tau(t))$ for system (2.16), one can recover the dependence of θ on τ, and hence, on r.

Since system (2.16) is π-periodic in θ, we will focus on the portion of its phase space for which $\theta \in [0, \pi]$ and $\tau \geq 0$.

2.2 Dynamics of (2.16) for $a = 0$.

System (2.16) for $a = 0$ is decoupled. In particular, the cylinder $\{\tau = 0\}$ is invariant and, on $\{\tau = 0\}$, system (2.16) is reduced to

$$\theta' = \frac{1}{f(\theta)}\rho, \quad \rho' = \frac{f_\theta(\theta)}{2f^2(\theta)}\rho^2 + K_1\frac{\sin(2\theta)}{2}. \tag{2.18}$$

Lemma 2.2. *System (2.18) is Hamiltonian with a Hamiltonian*

$$\mathcal{H} = \frac{1}{2f(\theta)}\rho^2 + \frac{K_1\cos(2\theta)}{4}. \tag{2.19}$$

Immediately, one has

Proposition 2.3. *The phase portrait of system (2.16) for $a = 0$ on $\{\tau = 0\}$ can be described as follows.*

(i) The equilibria $(0, 0, 0)$ and $(\pi, 0, 0)$ are saddles and there is a heteroclinic loop between $(0, 0, 0)$ and $(\pi, 0, 0)$. Both the heteroclinic orbits lie on the level set $\mathcal{H} = K_1/4$ of the Hamiltonian function $\mathcal{H}$.

(ii) The equilibrium $(\pi/2, 0, 0)$ is a center and it is surrounded by closed orbits between the heteroclinic loop.

(iii) Outside the heteroclinic loop, all orbits are periodic in θ (mod π) and ρ.

The phase portrait for $\tau > 0$ is determined by the product structure of system (2.16) for $a = 0$ so that each cylinder over an orbit on $\{\tau = 0\}$ described above is invariant. In particular, for any $\theta_0 \in [0, \pi/2]$, there are exactly two solutions of the boundary value problem (2.16) and (2.17) for $a = 0$, one with $\theta(t) \to 0$ as $t \to -\infty$ and the other with $\theta(t) \to \pi$ as $t \to -\infty$ (see Fig. 1).

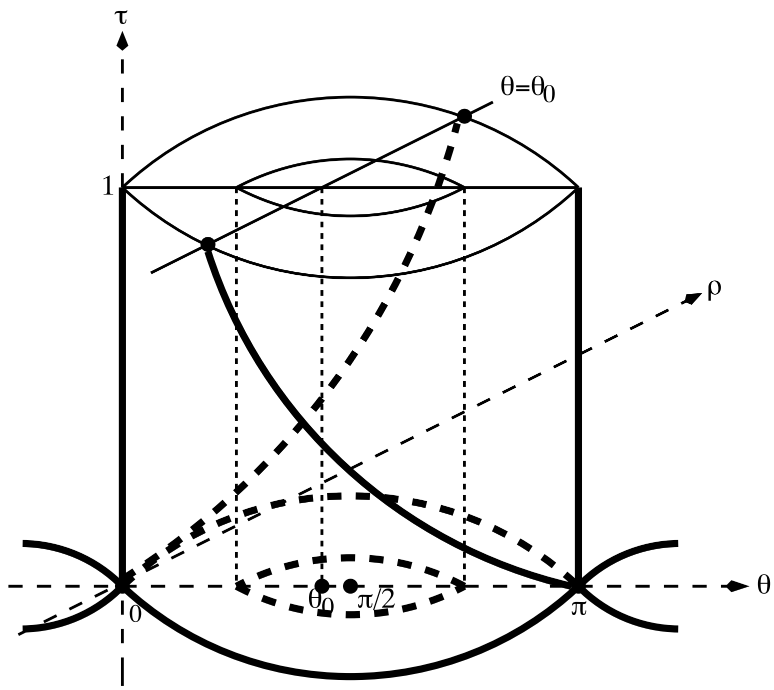

Figure 1: Thick curves are orbits of system (2.16). On $\{\tau = 0\}$, there are two heteroclinic orbits between $(0,0)$ and $(\pi, 0)$; insider the heteroclinic loop are periodic orbits centered about $(\pi/2, 0)$. Every cylinder over each orbit on $\{\tau = 0\}$ is invariant. For any $\theta_0 \in [0, \pi)$, there are two solutions symmetric to each other, one approaches $(0,0,0)$ and the other $(\pi, 0, 0)$ as $\tau \to 0$.

Note that, for the first solution with $\theta(t) \to 0$ as $t \to -\infty$, we have that $\rho^2 = K_1 s^2(\theta) f(\theta)$ or $f(\theta)(\theta')^2 = K_1 s^2(\theta)$. It then follows from $r = Re^t$ that

$$\frac{d\theta}{dr} = \frac{\sqrt{K_1} s(\theta)}{r\sqrt{f(\theta)}}.$$

Hence, from (2.12) with $\mathcal{E}(\theta_0) = \mathcal{E}(\theta(r;\theta_0))$,

$$\begin{aligned}\mathcal{E}(\theta_0) =& 2K_1 \int_0^R \frac{s^2(\theta)}{r}\, dr = 2\sqrt{K_1} \int_0^{\theta_0} s(\theta)\sqrt{f(\theta)} d\theta \qquad [\theta = \theta(r)] \\ =& 2\sqrt{K_1} \int_{c(\theta_0)}^1 \sqrt{K_3 + (K_1 - K_3) z^2}\, dz. \quad [z = c(\theta)]\end{aligned}$$

Note that the last integral can be evaluated explicitly in any cases of (K_1, K_3).

Since $c(\theta)$ is monotonically decreasing for $\theta \in [0, \pi/2]$, $\mathcal{E}(\theta_0)$ is monotonically increasing for $\theta_0 \in [0, \pi/2]$.

2.3 BVP of (2.16) and (2.17) for small $a > 0$.

We now consider the boundary value problem (2.16) and (2.17) for small $a > 0$, viewing it as a perturbation to the problem with $a = 0$. The product structure of system (2.16) for $a = 0$ can be treated as the unstable foliation over the invariant manifold $\{\tau = 0\}$. (It is the *unstable* foliation since $\tau' = \tau$.) The question is whether or not this foliation persists for $a > 0$. The general invariant manifold theory states that the foliation persists if and only if the invariant manifold $\{\tau = 0\}$ is *normally hyperbolic* ([10, 13, 17]). For this problem at hand, the normal hyperbolicity is reduced to check the eigenvalues of the linearization at $(0, 0, 0)$. One finds easily that the eigenvalues are -1, 1, and 1 with the second one in the direction normal to $\{\tau = 0\}$. In particular, $\{\tau = 0\}$ is not normally hyperbolic. The general theory would imply that the invariant manifold $\{\tau = 0\}$ and the foliation structure would be destroyed under *general perturbations*. On the other hand, system (2.16) with $a > 0$ is a *special perturbation* to that with $a = 0$. For example, for any a, $\{\tau = 0\}$ is *always* invariant. Will the foliation structure survive for small $a > 0$? It turns out it does !

2.3.1 Persistence of the foliations for small $a > 0$

We go back to (2.14) and make the scaling $r \to \xi^{1/3}$ which is not smooth only at $\xi = 0$. More precisely, we set

$$p(\xi) = \theta(\xi^{1/3}),\ q(\xi) = \rho(\xi^{1/3}).$$

With dot denoting the derivative with respect to ξ, we have, from (2.14),

$$\begin{aligned}
\dot{p} =& \frac{\xi^{-2/3}}{3\xi^{1/3} f(p)} q = \frac{1}{3\xi f(p)} q, \\
\dot{q} =& \frac{\xi^{-2/3} f'(p)}{6\xi^{1/3} f^2(p)} q^2 + K_1 \frac{\xi^{-2/3}\sin(2p)}{6\xi^{1/3}} - \frac{a(\gamma_1 + \gamma_2\cos(2p))\xi^{-2/3}\xi^{2/3}}{12 g(p)} \\
=& \frac{f'(p)}{6\xi f^2(p)} q^2 + K_1 \frac{\sin(2p)}{6\xi} - \frac{a(\gamma_1 + \gamma_2\cos(2p))}{12 g(p)}, \\
\dot{\xi} =& 1.
\end{aligned} \tag{2.20}$$

Rescale the independent variable to get

$$\begin{aligned}
\dot{p} =& \frac{1}{3f(p)} q, \\
\dot{q} =& \frac{f'(p)}{6f^2(p)} q^2 + K_1 \frac{\sin(2p)}{6} - \frac{a(\gamma_1 + \gamma_2\cos(2p))\xi}{12 g(p)}, \\
\dot{\xi} =& \xi.
\end{aligned} \tag{2.21}$$

Now the plane $\{\xi = 0\}$ is invariant and, on $\{\xi = 0\}$, the flow is again a Hamiltonian. The linearization at the saddle points $(k\pi, 0, 0)$ (identified with $(0, 0, 0)$) is

$$\begin{pmatrix} 0 & \frac{1}{3K_1} & 0 \\ \frac{K_1}{3} & 0 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues are $\pm 1/3$ and 1. Therefore, $\{\xi = 0\}$ is normally unstable, and hence, the unstable foliation persists for $a > 0$ small. This unstable foliation can be directly transformed to one for system (2.16) with small $a > 0$.

To this end, we make a remark concerning early works in [23, 24]. In these papers, the authors considered the last equation in (2.7) with a prescribed $u = u(r)$ for $r \in [0, R]$. The analysis in this paper applies to general $u(r)$ as long as $r^{-1}u'(r)$ is bounded as $r \to 0$, which is a physically reasonable requirement. In this case, system (2.14) becomes

$$\begin{aligned}\theta' =&\frac{1}{rf(\theta)}\rho,\\ \rho' =&\frac{f'(\theta)}{2rf^2(\theta)}\rho^2 + K_1\frac{\sin(2\theta)}{2r} + \frac{(\gamma_1+\gamma_2\cos(2\theta))ru'(r)}{2}.\end{aligned} \tag{2.22}$$

The rescaling

$$p(\xi) = \theta(\xi^{1/3}),\ q(\xi) = \rho(\xi^{1/3})$$

then converts (2.22) to an autonomous system augmenting $\dot{\xi} = 1$,

$$\begin{aligned}\dot{p} =&\frac{\xi^{-2/3}}{3\xi^{1/3}f(p)}q = \frac{1}{3\xi f(p)}q,\\ \dot{q} =&\frac{f'(p)}{6\xi f^2(p)}q^2 + K_1\frac{\sin(2p)}{6\xi} + \frac{(\gamma_1+\gamma_2\cos(2p))\xi^{-1/3}u'(\xi^{1/3})}{6},\\ \dot{\xi} =&1.\end{aligned} \tag{2.23}$$

One more rescaling gives

$$\begin{aligned}\dot{p} =&\frac{1}{3f(p)}q,\\ \dot{q} =&\frac{f'(p)}{6f^2(p)}q^2 + K_1\frac{\sin(2p)}{6} + \frac{(\gamma_1+\gamma_2\cos(2p))\xi^{2/3}u'(\xi^{1/3})}{6},\\ \dot{\xi} =&\xi.\end{aligned} \tag{2.24}$$

One can check easily that all arguments in previous sections work for this system as long as $r^{-1}u'(r)$ is bounded as $r \to 0$ (It implies that the perturbation term is Lipschitz). Note that $u'(r)$ is allowed to change sign.

2.3.2 Results on the boundary problem (2.14) and (2.15).

Based on the above analysis, we now summarize the result on the boundary problem (2.14) and (2.15) for $a \neq 0$ but near zero.

Theorem 2.4. *Fix $a \geq 0$ small. For any $\theta_0 \in [0, \pi/2]$, there are exactly two solutions $(\theta_1(r;\theta_0), \rho_1(r;\theta_0))$ and $(\theta_2(r;\theta_0), \rho_2(r;\theta_0))$ of the boundary value problem (2.14) and (2.15) with $\theta_1(r;\theta_0) \to 0$ and $\theta_2(r;\theta_0) \to \pi$ as $r \to 0$. Furthermore, for any $s \in \mathbb{R}_+$, there exists a unique $\theta_0 = \theta_0(s) \in [0, \pi/2]$ so that $\theta_1'(r;\theta_0) \to s$ and $\theta_2'(r;\theta_0) \to -s$ as $r \to 0$.*

The latter statement of Theorem 2.4 follows from the following observation on system (2.16). Since all solutions of (2.16) that approach $(0,0,0)$ as $t \to -\infty$ lie on the unstable manifold W^u of $(0,0,0)$ and the unstable eigenvalues of $(0,0,0)$ is a *proper* node on W^u for the *linearization* (due to the repeated eigenvalue 1). It is a standard result (see, e.g., p.143 in [20]) that $(0,0,0)$ is also a *proper* node on W^u for the *nonlinear* system (2.16) as long as the system on W^u is $\mathcal{C}^2$. Therefore, any direction along which an orbit approaches $(0,0,0)$ as $t \to -\infty$ can be realized. This converts directly to the last statement.

3 Apparent viscosity

The purpose of this section is to provide an analytical procedure for the Taylor expansion of the *apparent viscosity* η for Poiseuille flow of nematic liquid-crystals. We start with a brief recall of the concept of apparent viscosity, Hagen-Poiseuille law and Atkin's result for liquid-crystals.

3.1 Hagen-Poiseuille law and Atkin's result

When two layers of liquid in contact with each other move at different speeds in x-direction, there will be a force between them. This force F is proportional to the area A of contact and the velocity difference in the direction of flow du/dy. The *apparent viscosity* η is defined to be the proportionality constant; that is, the force on the fast layer is

$$F = -\eta A \frac{du}{dy}.$$

For liquid flow along a tube, under some conditions such as the flow is laminar and non-turbulent, Poiseuille and Hagen independently derived the following law – *Hagen-Poiseuille law* – for apparent viscosity:

$$\eta = \frac{\pi a R^4}{8Q} \tag{3.25}$$

where a is the pressure gradient along the tube, R is the radius of the tube and Q is the volume flux rate given by

$$Q = \int_0^{2\pi} \int_0^R r u(r, \phi) dr d\phi.$$

In ([2]), Atkin extended the scaling argument of Ericksen for shear flows of liquid-crystals ([8]) to the case of Poiseuille flow. Under reasonable assumptions, Atkin showed that the quantity Q/R is a function of aR^3, or equivalently, aR^3 is a function of Q/R. It then follows from Hagen-Poiseuille law (3.25) that the apparent viscosity η is a function of aR^3 only or a function of Q/R only:

$$\eta = \mathcal{S}_1\left(\frac{Q}{R}\right) \quad \text{and} \quad \eta = \mathcal{S}_2(aR^3)$$

for some functions $\mathcal{S}_1$ and $\mathcal{S}_2$.

As discussed in the introduction, this result is remarkable for at least two reasons: first of all, it is justified by the experiments of Fisher and Fredrickson [12] and hence provides a strong support to the Ericksen-Leslie continuum theory for liquid-crystals; secondly, it reveals a significant difference between the liquid-crystals and isotropic fluid: for the latter, Coleman and Noll ([4]) showed that aR is a function of Q/R^3, or equivalently, Q/R^3 is a function of aR, which imply that the apparent viscosity $\eta = \tilde{\mathcal{S}}_1(Q/R^3)$ or $\eta = \tilde{\mathcal{S}}_2(aR)$ for some functions $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$.

Tseng, Silver and Finlayson ([22]) have solved system (2.10) numerically with boundary condition $\theta_0 = -\pi/2$ for a couple of sets of parameters and the results match well with the experimental data of Fisher and Fredrickson ([12], see also §3.6.3 in [3]).

3.2 Atkin's result for radially symmetric flow

We will derive a direct dependence of Q/R on aR^3 for Poiseuille flow of liquid-crystals which will be the starting point of a procedure for the Taylor expansion for $\eta = \mathcal{S}(Q/R)$.

For Poiseuille flow of liquid-crystals, we recall, from (2.13), that

$$u(r; a, R) = \frac{a}{2}\int_r^R \frac{s}{g(\theta(s; a, R))}ds,$$

where $\theta(r; a, R)$ is the solution of (2.14) and (2.15). Thus, with substitutions indicated inside brackets between integrals,

$$\begin{aligned}
Q =&\pi a\int_0^R r\int_r^R \frac{s}{g(\theta(s; a, R))}dsdr \qquad [r = Rp]\\
=&\pi aR^2\int_0^1 p\int_{Rp}^R \frac{s}{g(\theta(s; a, R))}dsdp \qquad [s = Rz]\\
=&\pi aR^4\int_0^1 p\int_p^1 \frac{z}{g(\theta(Rz; a, R))}dzdp \qquad [\text{change integration order}]\\
=&\pi aR^4\int_0^1 \frac{z}{g(\theta(Rz; a, R))}\int_0^z pdp\, dz = \frac{\pi aR^4}{2}\int_0^1 \frac{z^3}{g(\theta(Rz; a, R))}dz.
\end{aligned} \tag{3.26}$$

Next, we rescale $r = Re^t$, set $\phi(t) = \theta(Re^t)$ and $\psi(t) = \rho(Re^t)$, and introduce $\tau = r/R$. Then system (2.14) becomes

$$\begin{aligned}
\phi' =&\frac{1}{f(\phi)}\psi,\\
\psi' =&\frac{f_\phi(\phi)}{2f^2(\phi)}\psi^2 + K_1\frac{\sin(2\phi)}{2} - \varepsilon\tau^3 G(\phi),\\
\tau' =&\tau.
\end{aligned} \tag{3.27}$$

In (3.27), for briefness, we have introduced

$$\varepsilon = aR^3, \quad G(\phi) = \frac{\gamma_1 + \gamma_2 \cos(2\phi)}{4g(\phi)}. \tag{3.28}$$

Note that, for bounded $\frac{d\theta}{dr}(0^+)$, it follows from $r = Re^t$ and $\phi(t) = \theta(r)$ that

$$\frac{d\phi}{dt}(t) = \frac{d\theta}{dr}(r)\frac{dr}{dt} = Re^t \frac{d\theta}{dr}(r) \to 0 \text{ as } t \to -\infty.$$

So the boundary condition (2.15) is now

$$\tau(0) = 1, \quad \phi(0) = \theta_0, \quad (\phi(t), \psi(t)) \to 0 \text{ as } t \to -\infty. \tag{3.29}$$

Thus the flux Q in (3.26) becomes

$$Q = \frac{\pi a R^4}{2} \int_0^1 \frac{z^3}{g(\theta(Rz; a, R))} dz = \frac{\pi a R^4}{2} \int_{-\infty}^0 \frac{e^{4t}}{g(\phi(t; a, R))} dt \qquad [z = e^t]$$

where $\phi(t; a, R)$ is the ϕ-component of the solution of problem (3.27) and (3.29).

In view of (3.27) and (3.29), $\phi(t; a, R)$ actually depends on $\varepsilon = aR^3$ so we write $\phi(t; aR^3)$ instead of $\phi(t; a, R)$ in the sequel, and hence,

$$\frac{Q}{R} = \frac{\pi a R^3}{2} \mathcal{L}(aR^3), \quad \text{where } \mathcal{L}(aR^3) = \int_{-\infty}^0 \frac{e^{4t}}{g(\phi(t; aR^3))} dt \tag{3.30}$$

is a function of $\varepsilon = aR^3$. This is simply Atkin's result for this case.

In the rest of this section, we will first provide an analytic procedure for the Taylor expansion of the function $\mathcal{L}(\varepsilon)$ at $\varepsilon = 0$ and then use the relation (3.30) to obtain Taylor expansions for $\eta = \mathcal{S}_1(Q/R)$ and $\eta = \mathcal{S}_2(aR^3)$.

3.3 Taylor expansion approximations of η.

For $\theta_0 \in [0, \pi/2]$, we will consider the first solution in Theorem 2.4. For $\varepsilon \geq 0$, let $(\phi(t; \varepsilon), \psi(t; \varepsilon), \tau(t; \varepsilon))$ be the unique solution of (3.27) satisfying (3.29). Since it is obtained by the intersection of the unstable manifold of the origin with the line $\{(\phi, \psi, \tau) : \theta = \theta_0, \ \tau = 0\}$ and the unstable manifold is smooth in ε, $(\phi(t; \varepsilon), \psi(t; \varepsilon), \tau(t; \varepsilon))$ is smooth in ε. Therefore, we can proceed to find a Taylor expansion of $(\phi(t; \varepsilon), \psi(t; \varepsilon))$ in ϵ and, in turn, a Taylor expansion of $\mathcal{L}(\varepsilon)$ defined in (3.30). Let

$$\phi(t; \varepsilon) = \sum_{j=0} \varepsilon^j \phi_j(t), \quad \psi(t; \varepsilon) = \sum_{j=0} \varepsilon^j \psi_j(t), \quad \tau(t; \varepsilon) = \tau(t) \tag{3.31}$$

with $\phi(0; \varepsilon) = \theta_0 \in [0, \pi/2)$, $\tau(0) = 1$ and $(\phi(t, \varepsilon), \psi(t, \varepsilon)) \to 0$ as $t \to -\infty$. In particular, $\phi_0(0) = \theta_0$ and $\phi_j(0) = 0$ for $j \geq 1$. It turns out the case θ_0 needs to be treated separably due to a reason to be discussed below.

Note that $\tau(t;\varepsilon)=\tau(t)=e^t$. So we only need to slove (ϕ_j,ψ_j)'s. Substitute (3.31) into (3.27) to get,

$$\begin{aligned}&\phi_0'=\frac{1}{f(\phi_0)}\psi_0,\quad \psi_0'=\frac{f_\phi(\phi_0)}{2f^2(\phi_0)}\psi_0^2+\frac{K_1}{2}\sin(2\phi_0);\\&\phi_0(0)=\theta_0,\quad (\phi_0(t),\psi_0(t))\to 0 \ \text{ as } \ t\to-\infty\end{aligned}\tag{3.32}$$

and, for $j\geq 1$,

$$\begin{aligned}&\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix}'=A(t)\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix}+M_j(t),\\&\phi_j(0)=0,\quad (\phi_j(t),\psi_j(t))\to 0 \ \text{ as } \ t\to-\infty\end{aligned}\tag{3.33}$$

where $A(t)$ is the linearization at $(\phi_0(t),\psi_0(t))$ of the vector field in (3.32)

$$\begin{pmatrix}-\frac{f_\phi(\phi_0(t))}{f^2(\phi_0(t))}\psi_0(t) & \frac{1}{f(\phi_0(t))}\\ \frac{f(\phi_0(t))f_{\phi\phi}(\phi_0(t))-2f_\phi^2(\phi_0(t))}{2f^3(\phi_0(t))}\psi_0^2(t)+K_1\cos(2\phi_0(t)) & \frac{f_\phi(\phi_0(t))}{f^2(\phi_0(t))}\psi_0(t)\end{pmatrix},$$

and $M_j(t)=(m_{j1}(t),m_{j2}(t))^T$ depends on t, $\phi_k(t)$, $\psi_k(t)$ for $k<j$.

3.3.1 Zeroth order system (3.32) and its linearization

As we know, the (ϕ_0,ψ_0)-system of (3.32) has the following integral

$$\mathcal{H}=\frac{1}{2f(\phi_0)}\psi_0^2+\frac{K_1\cos(2\phi_0)}{4}.$$

In particular, the orbit $(\phi_0(t),\psi_0(t))$ will be on the level $\mathcal{H}(0,0)=K_1/4$ with nonnegative ψ_0, and we have

$$\psi_0(t)=\sqrt{K_1f(\phi_0(t))}\sin\phi_0(t),\quad \phi_0'(t)=\frac{\sqrt{K_1}\sin\phi_0(t)}{\sqrt{f(\phi_0(t))}}.$$

If $\theta_0=0$, then $\phi_0(t)=\psi_0(t)=0$.

The homogeneous part of (3.33)

$$x'=A(t)x\tag{3.34}$$

is the linearization of (3.32) along $(\phi_0(t),\psi_0(t))$. Therefore, if $x(t)$ is a solution of (3.34), then

$$\langle\nabla\mathcal{H}(\phi_0(t),\psi_0(t)),x(t)\rangle=\langle\nabla\mathcal{H}(\phi_0(0),\psi_0(0)),x(0)\rangle$$

for all t (see, e.g., Proposition 2.2 in [1]). Using this fact, one finds

Proposition 3.1. *The principal fundamental matrix solution of (3.34) at $t = 0$ is given by*

$$Z(t) = \begin{pmatrix} \Phi_1(t) & \Phi_2(t) \\ \Psi_1(t) & \Psi_2(t) \end{pmatrix},$$

where

(i) for $\theta_0 = 0$ (in this case $A(t)$ is a constant matrix),

$$\Phi_1(t) = \Psi_2(t) = \cosh(t),\ \Phi_2(t) = \frac{1}{K_1}\sinh(t),\ \Psi_1(t) = K_1 \sinh(t); \quad (3.35)$$

(ii) for $\theta_0 \in (0, \pi/2]$,

$$\begin{aligned} \Phi_1(t) =& \frac{\phi_0'(t)}{\phi_0'(0)} - \frac{\psi_0'(0)}{\phi_0'(0)}\Phi_2(t),\ \Phi_2(t) = \frac{\phi_0'(0)\phi_0'(t)}{K_1^{3/2}} \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy \\ \Psi_1(t) =& \frac{\psi_0'(t)}{\phi_0'(t)}\Phi_1(t) - \frac{\psi_0'(0)}{\phi_0'(t)},\ \Psi_2(t) = \frac{\psi_0'(t)}{\phi_0'(t)}\Phi_2(t) + \frac{\phi_0'(0)}{\phi_0'(t)} \end{aligned} \quad (3.36)$$

with

$$\psi_0'(0) = \frac{K_1 f_\phi(\theta_0)}{2f(\theta_0)} \sin^2\theta_0 + K_1 \sin\theta_0 \cos\theta_0, \quad \phi_0'(0) = \frac{\sqrt{K_1}\sin\theta_0}{\sqrt{f(\theta_0)}}.$$

Note that

$$\frac{\psi_0'(0)}{\phi_0'(0)}\Psi_2(t) + \Psi_1(t) = \frac{\psi_0'(t)}{\phi_0'(t)}\left(\frac{\psi_0'(0)}{\phi_0'(0)}\Phi_2(t) + \Phi_1(t)\right) = \frac{\psi_0'(t)}{\phi_0'(0)}. \quad (3.37)$$

Furthermore, we have

Proposition 3.2. *System (3.34) has an exponential dichotomy (ED) for $t \in (-\infty, 0)$ with a choice of stable and unstable projections:*

(i) for $\theta_0 = 0$,

$$P = \begin{pmatrix} 0 & 0 \\ -K_1 & 1 \end{pmatrix} \ \text{and}\ Q = I - P = \begin{pmatrix} 1 & 0 \\ K_1 & 0 \end{pmatrix};$$

(ii) for $\theta_0 \in (0, \pi/2]$,

$$P = \begin{pmatrix} 0 & 0 \\ -\frac{\psi_0'(0)}{\phi_0'(0)} & 1 \end{pmatrix} \ \text{and}\ Q = I - P = \begin{pmatrix} 1 & 0 \\ \frac{\psi_0'(0)}{\phi_0'(0)} & 0 \end{pmatrix}.$$

More precisely, there exist $K > 0$ and $\gamma > 0$ such that

$$\begin{aligned} \|Z(t)PZ^{-1}(s)\| \leq& K e^{-\gamma(t-s)} \ \text{for}\ 0 \geq t \geq s; \\ \|Z(t)QZ^{-1}(s)\| \leq& K e^{\gamma(t-s)} \ \text{for}\ t \leq s \leq 0. \end{aligned}$$

Proof. Since $(\phi_0(t), \psi_0(t)) \to (0,0)$ as $t \to -\infty$ and $(0,0)$ is a saddle point, we conclude that system (3.34) admits an ED over $(-\infty, 0)$. For a pair of projections, we only need the image of the unstable project Q to be the span of the unstable direction which is $(1, K_1)^T$ for $\theta_0 = 0$ and $(\phi_0'(0), \psi_0'(0))^T$ for $\theta_0 \in (0, \pi/2]$. The projection Q does the job with the compliment project P on the span of $(0,1)^T$. □

3.3.2 Higher order solutions $(\phi_j(t), \psi_j(t))$ for $j \geq 1$.

It is a standard result that once the homogeneous system (3.34) has an ED over $(-\infty, 0)$, there is a unique solution of (3.33) bounded over $(-\infty, 0)$.

Proposition 3.3. *The solution $(\phi_j(t), \psi_j(t))$ of (3.33) that is bounded for $t \in (-\infty, 0)$ with $(\phi_j(0), \psi_j(0)) = (0, b_j)$ is given by:*

(i) for $\theta_0 = 0$,

$$\begin{aligned} \begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} =& \frac{e^t}{K_1} \int_0^t (K_1 \cosh(s) m_{j1}(s) - \sinh(s) m_{j2}(s))\, ds \begin{pmatrix} 1 \\ K_1 \end{pmatrix} \\ &+ \frac{1}{K_1} \int_{-\infty}^t e^s \big(m_{j2}(s) - K_1 m_{j1}(s)\big) ds \begin{pmatrix} \sinh(t) \\ K_1 \cosh(t) \end{pmatrix}, \end{aligned} \tag{3.38}$$

in particular,

$$\begin{aligned} \phi_j(t) =& \frac{e^t}{K_1} \int_0^t (K_1 \cosh(s) m_{j1}(s) - \sinh(s) m_{j2}(s))\, ds \\ &+ \frac{\sinh(t)}{K_1} \int_{-\infty}^t e^s \big(m_{j2}(s) - K_1 m_{j1}(s)\big) ds; \end{aligned} \tag{3.39}$$

(ii) for $\theta_0 \in (0, \pi/2]$,

$$\begin{aligned} \begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} =& K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_0^t \phi_0'(s) N_j(s, \phi_0(s)) \int_{\phi_0(s)}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy ds \\ &+ K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{-\infty}^0 \phi_0'(s) N_j(s, \phi_0(s)) ds \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy \\ &+ \begin{pmatrix} 0 \\ \frac{1}{\phi_0'(t)} \end{pmatrix} \int_{-\infty}^t \phi_0'(s) N_j(s, \phi_0(s)) ds, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} N_j(s, \phi_0(s)) =& m_{j2}(s) - \frac{\psi_0'(s)}{\phi_0'(s)} m_{j1}(s), \quad \textit{or} \\ N_j(s, y) =& m_{j2}(s) - \sqrt{K_1 f(y)} \sin y \left(\frac{f_\phi(y)}{2 f(y)} + \frac{\cos y}{\sin y} \right) m_{j1}(s), \\ \psi_j(t) =& \frac{\psi_0'(t)}{\phi_0'(t)} \phi_j(t) + \frac{1}{\phi_0'(t)} \int_{-\infty}^t \phi_0'(s) N_j(s, \phi_0(s)) ds. \end{aligned} \tag{3.41}$$

Proof. We prove the statement for $\theta_0 \in (0, \pi/2]$. The case with $\theta_0 = 0$ follows the same line of proof. The solution of (3.33) with $(\phi(0), \psi(0)) = (0, b_j)$ is

$$\begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} = Z(t) \begin{pmatrix} 0 \\ b_j \end{pmatrix} + Z(t) \int_0^t Z^{-1}(s) M_j(s) ds. \tag{3.42}$$

Apply $PZ^{-1}(t)$ to (3.42) to get

$$\begin{aligned} PZ^{-1}(t) \begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} &= P \begin{pmatrix} 0 \\ b_j \end{pmatrix} + \int_0^t PZ^{-1}(s) M_j(s) ds \\ &= \begin{pmatrix} 0 \\ b_j \end{pmatrix} + \int_0^t PZ^{-1}(s) M_j(s) ds = \begin{pmatrix} 0 \\ b_j + I(0,t) \end{pmatrix}, \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} I(r,t) &= \int_r^t \left(\left(\frac{\psi_0'(0)}{\phi_0'(0)} \Phi_2(s) + \Phi_1(s) \right) m_{j2}(s) - \left(\frac{\psi_0'(0)}{\phi_0'(0)} \Psi_2(s) + \Psi_1(s) \right) m_{j1}(s) \right) ds \\ &= \int_r^t \left(\frac{\phi_0'(s)}{\phi_0'(0)} m_{j2}(s) - \frac{\psi_0'(s)}{\phi_0'(0)} m_{j1}(s) \right) ds \qquad \text{[Used (3.36), (3.37)]} \\ &= \frac{1}{\phi_0'(0)} \int_r^t \left(\phi_0'(s) m_{j2}(s) - \psi_0'(s) m_{j1}(s) \right) ds. \end{aligned}$$

We note that

$$\frac{\psi_0'(s)}{\phi_0'(s)} = \sqrt{K_1 f(\phi_0(s))} \sin \phi_0(s) \left(\frac{f_\phi(\phi_0(s))}{2f(\phi_0(s))} + \frac{\cos \phi_0(s)}{\sin \phi_0(s)} \right).$$

Taking $t \to -\infty$ in (3.43), we get

$$\begin{pmatrix} 0 \\ b_j \end{pmatrix} = -\int_0^{-\infty} PZ^{-1}(s) M_j(s) ds = \int_{-\infty}^0 PZ^{-1}(s) M_j(s) ds;$$

in particular, $b_j = -I(0, -\infty) = I(-\infty, 0)$. Substitute back to (3.42) to get

$$
\begin{aligned}
\begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} =& Z(t)\int_{-\infty}^{0} PZ^{-1}(s)M_j(s)ds + Z(t)\int_0^t PZ^{-1}(s)M_j(s)ds \\
&+ Z(t)\int_0^t QZ^{-1}(s)M_j(s)ds \\
=& Z(t)\int_0^t QZ^{-1}(s)M_j(s)ds + Z(t)\int_{-\infty}^{t} PZ^{-1}(s)M_j(s)ds \\
=& \begin{pmatrix} \Phi_1(t) + \frac{\psi_0'(0)}{\phi_0'(0)}\Phi_2(t) \\ \Psi_1(t) + \frac{\psi_0'(0)}{\phi_0'(0)}\Psi_2(t) \end{pmatrix} \int_0^t \left(\Psi_2(s)m_{j1}(s) - \Phi_2(s)m_{j2}(s)\right) ds \\
&+ \begin{pmatrix} \Phi_2(t) \\ \Psi_2(t) \end{pmatrix} I(-\infty, t) \\
=& \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \frac{1}{\phi_0'(0)} \int_0^t \left(\Psi_2(s)m_{j1}(s) - \Phi_2(s)m_{j2}(s)\right) ds \\
&+ \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \frac{\Phi_2(t)}{\phi_0'(t)} I(-\infty, 0) + \begin{pmatrix} 0 \\ \frac{\phi_0'(0)}{\phi_0'(t)} \end{pmatrix} I(-\infty, t) \\
=& - K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_0^t \left(\phi_0'(s)m_{j2}(s) - \psi_0'(s)m_{j1}(s)\right) \int_{\theta_0}^{\phi_0(s)} \frac{\sqrt{f(y)}}{\sin^3 y} dyds \\
&+ K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{-\infty}^{t} \left(\phi_0'(s)m_{j2}(s) - \psi_0'(s)m_{j1}(s)\right) ds \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy \\
&+ \begin{pmatrix} 0 \\ \frac{1}{\phi_0'(t)} \end{pmatrix} \int_{-\infty}^{t} \left(\phi_0'(s)m_{j2}(s) - \psi_0'(s)m_{j1}(s)\right) ds \\
=& K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_0^t \phi_0'(s)N_j(s,\phi_0(s)) \int_{\phi_0(s)}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dyds \\
&+ K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{-\infty}^{0} \phi_0'(s)N_j(s,\phi_0(s))ds \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy \\
&+ \begin{pmatrix} 0 \\ \frac{1}{\phi_0'(t)} \end{pmatrix} \int_{-\infty}^{t} \phi_0'(s)N_j(s,\phi_0(s))ds.
\end{aligned}
$$

This completes the proof. □

Next result does not work for $\theta_0 = 0$ since $\phi_0(t) = \psi_0(t) = 0$ for this case.

Corollary 3.4. *Let $\theta_0 \in (0, \pi/2]$. For each $j \geq 1$, $\phi_j(t) = \mathcal{F}_j(\phi_0(t))$ where*

$$
\mathcal{F}_j(x) = \frac{\sin x}{K_1\sqrt{f(x)}} \int_{\theta_0}^{x} \frac{\sqrt{f(z)}}{\sin^3 z} \int_0^z N_j(T(y), y)dydz
$$

with N_j given in (3.41) and

$$
T(y) = \int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz.
$$

Proof. With N_j given in (3.41) and

$$T(y) = \int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz,$$

we have

$$\begin{aligned}
\begin{pmatrix} \phi_j(t) \\ \psi_j(t) \end{pmatrix} =& K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{\theta_0}^{\phi_0(t)} N_j(T(y), y) \int_{y}^{\phi_0(t)} \frac{\sqrt{f(z)}}{\sin^3 z} dz dy \\
&+ K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{0}^{\theta_0} N_j(T(y), y) dy \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(y)}}{\sin^3 y} dy \\
&+ \begin{pmatrix} 0 \\ \frac{1}{\phi_0'(t)} \end{pmatrix} \int_{0}^{\phi_0(t)} N_j(T(y), y) dy \qquad (3.44) \\
=& K_1^{-3/2} \begin{pmatrix} \phi_0'(t) \\ \psi_0'(t) \end{pmatrix} \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(z)}}{\sin^3 z} \int_{0}^{z} N_j(T(y), y) dy dz \\
&+ \begin{pmatrix} 0 \\ \frac{1}{\phi_0'(t)} \end{pmatrix} \int_{0}^{\phi_0(t)} N_j(T(y), y) dy.
\end{aligned}$$

In particular,

$$\phi_j(t) = \frac{\sin \phi_0(t)}{K_1 \sqrt{f(\phi_0(t))}} \int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(z)}}{\sin^3 z} \int_{0}^{z} N_j(T(y), y) dy dz.$$

This completes the proof. □

3.3.3 The expansions of $\mathcal{L}$ and η.

For the function g in (2.8) and for an expansion $y(\varepsilon) = \varepsilon^j y_j$, denote

$$\frac{1}{g(y(\varepsilon))} = \varepsilon^j C_j(y_0, \cdots, y_j) = C_0(y_0) + \varepsilon C_1(y_0, y_1) + \cdots .$$

We note that $C_0(y_0) = 1/g(y_0)$. For $\theta_0 \in (0, \pi/2]$, we also set, for each $j \geq 0$,

$$D_j(y_0) = C_j(y_0, \mathcal{F}_1(y_0), \cdots, \mathcal{F}_j(y_0)),$$

where $\mathcal{F}_j$'s are defined in Corollary 3.4. We then have

Proposition 3.5. *The asymptotic expansion of $\mathcal{L}$ is*

$$\mathcal{L}(\varepsilon) = \varepsilon^j L_j = L_0 + \varepsilon L_1 + \cdots ,$$

where (i) for $\theta_0 = 0$,

$$L_j = \int_{-\infty}^{0} e^{4t} C_j(\phi_0(t), \cdots, \phi_j(t)) dt,$$

where $\phi_j(t)$'s are defined in (3.39); in particular,

$$L_0 = \frac{1}{2(\alpha_3 + \alpha_6 + \alpha_4)} > 0;$$

(ii) for $\theta_0 \in (0, \pi/2]$,

$$L_j = \int_0^{\theta_0} \frac{\sqrt{f(y)}}{\sqrt{K_1}\sin y} \exp\left\{4\int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz\right\} D_j(y)dy,$$

in particular, $L_0 > 0$.

Proof. (i) For $\theta_0 = 0$, by the definition of C_j, we have

$$\frac{1}{g(\phi(t;\varepsilon))} = \varepsilon^j C_j(\phi_0(t), \cdots, \phi_j(t)).$$

Thus, from (3.30),

$$\mathcal{L}(\varepsilon) = \int_{-\infty}^{0} \frac{e^{4t}}{g(\phi(t;\varepsilon))} dt = \varepsilon^j \int_{-\infty}^{0} e^{4t} C_j(\phi_0(t), \cdots, \phi_j(t)) dt;$$

In particular,

$$L_0 = \int_{-\infty}^{0} e^{4t} C_0(\phi_0(t)) dt = \frac{1}{g(0)} \int_{-\infty}^{0} e^{4t} dt = \frac{1}{2(\alpha_3 + \alpha_6 + \alpha_4)} > 0.$$

The latter inequality is a consequence of (1.6).

(ii) For $\theta_0 \in (0, \pi/2]$, by the definition of D_j, we have

$$\frac{1}{g(\phi(t;\varepsilon))} = \varepsilon^j D_j(\phi_0(t)).$$

Note that

$$e^t = \tau = \exp\left\{\int_{\theta_0}^{\phi_0(t)} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz\right\}, \ \phi_0'(t) = \sqrt{\frac{K_1}{f(\phi_0(t))}} \sin(\phi_0(t)).$$

Thus, from (3.30),

$$\begin{aligned}\mathcal{L}(\varepsilon) = &\int_{-\infty}^{0} \frac{e^{4t}}{g(\phi(t;\varepsilon))} dt = \varepsilon^j \int_{-\infty}^{0} e^{4t} D_j(\phi_0(t)) dt \\ =&\varepsilon^j \int_0^{\theta_0} \frac{\sqrt{f(y)}}{\sqrt{K_1}\sin y} \exp\left\{4\int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz\right\} D_j(y)dy.\end{aligned}$$

We have used the change of variable $y = \phi_0(t)$ in the last step. In particular,

$$L_0 = \int_0^{\theta_0} \frac{\sqrt{f(y)}}{\sqrt{K_1}\sin y} \exp\left\{4\int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz\right\} D_0(y)dy$$
$$= \int_0^{\theta_0} \frac{\sqrt{f(y)}}{\sqrt{K_1} g(y)\sin y} \exp\left\{4\int_{\theta_0}^{y} \frac{\sqrt{f(z)}}{\sqrt{K_1}\sin z} dz\right\} dy > 0$$

This completes the proof. $\square$

For any integers $n \geq k \geq 1$, denote

$$\mathbb{N}_+^k(n) = \left\{\sigma = (\sigma_1, \cdots, \sigma_k) \in \mathbb{N}_+^k : \ \sigma_j \geq 1, \ \sigma_1 + \cdots + \sigma_k = n\right\}.$$

Let U_j's be defined recursively via

$$\frac{\pi}{2} L_0 U_0 = 1, \quad \sum_{k=1}^{n} L_{k-1} \sum_{\sigma \in \mathbb{N}_+^k(n)} U_{\sigma_1 - 1} \cdots U_{\sigma_k - 1} = 0 \ \text{ for } \ n > 1. \tag{3.45}$$

Since $L_0 > 0$ from Proposition 3.5, the above formula defines U_j's uniquely. We have

Theorem 3.6. *In terms of Q/R, the apparent viscosity is given by*

$$\eta = \mathcal{S}_1\left(\frac{Q}{R}\right) = \frac{\pi}{8}\sum_{i\geq 0} U_i \left(\frac{Q}{R}\right)^i$$

where U_i's are determined by (3.45).

Proof. If we set

$$aR^3 = \sum_{i\geq 1} U_{i-1} \left(\frac{Q}{R}\right)^i,$$

then

$$\eta = \mathcal{S}_1\left(\frac{Q}{R}\right) = \frac{\pi}{8}\sum_{i\geq 1} U_{i-1} \left(\frac{Q}{R}\right)^{i-1} = \frac{\pi}{8}\sum_{i\geq 0} U_i \left(\frac{Q}{R}\right)^i.$$

On the other hand, it follows from (3.30) and Proposition 3.5 that

$$
\begin{aligned}
\frac{Q}{R} =& \frac{\pi}{2}\sum_{k\geq 1} L_{k-1}(aR^3)^k \\
=& \frac{\pi}{2}\sum_{k\geq 1} L_{k-1}\left(\sum_{i\geq 1} U_{i-1}\left(\frac{Q}{R}\right)^i\right)^k \\
=& \frac{\pi}{2}\sum_{k\geq 1} L_{k-1}\sum_{n\geq k}\sum_{\sigma\in\mathbb{N}_+^k(n)} U_{\sigma_1-1}\left(\frac{Q}{R}\right)^{\sigma_1}\cdots U_{\sigma_k-1}\left(\frac{Q}{R}\right)^{\sigma_k} \\
=& \frac{\pi}{2}\sum_{k\geq 1} L_{k-1}\sum_{n\geq k}\left(\frac{Q}{R}\right)^n\sum_{\sigma\in\mathbb{N}_+^k(n)} U_{\sigma_1-1}\cdots U_{\sigma_k-1} \\
=& \frac{\pi}{2}\sum_{n\geq 1}\left(\frac{Q}{R}\right)^n\sum_{k=1}^{n} L_{k-1}\sum_{\sigma\in\mathbb{N}_+^k(n)} U_{\sigma_1-1}\cdots U_{\sigma_k-1}.
\end{aligned}
$$

Therefore, U_j's are determined by (3.45). □

Let V_j's be defined recursively via

$$
V_0L_0 = 1, \quad \sum_{j=0}^{n} V_{n-j}L_j = 0 \ \text{ for } \ n\geq 1, \tag{3.46}
$$

or,

$$
V_j = (-1)^j\frac{1}{L_0^{j+1}}\det N_j(L_0, L_1, \cdots, L_j), \tag{3.47}
$$

where

$$
N_j = N_j(L_0, L_1, \cdots, L_j) = \begin{pmatrix}
L_1 & L_0 & 0 & 0 & \cdots & 0 \\
L_2 & L_1 & L_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
L_{j-2} & L_{j-1} & \cdots & L_1 & L_0 & 0 \\
L_{j-1} & L_{j-2} & \cdots & L_2 & L_1 & L_0 \\
L_j & L_{j-1} & \cdots & L_3 & L_2 & L_1
\end{pmatrix}.
$$

We have

Theorem 3.7. *In terms of aR^3, the apparent viscosity η has the expansion*

$$
\eta = \mathcal{S}_2(aR^3) = \frac{1}{4}\sum_{j\geq 0} V_j(aR^3)^j
$$

where V_j's are determined by (3.46).

Proof. If we set

$$\frac{1}{\mathcal{L}(\varepsilon)} = \sum_{j\geq 0} V_j \varepsilon^j,$$

then

$$\eta = \frac{\pi a R^4}{8Q} = \frac{1}{4\mathcal{L}(aR^3)} = \frac{1}{4}\sum_{j\geq 0} V_j (aR^3)^j.$$

On the other hand, we have

$$1 = \mathcal{L}(\varepsilon)\sum_{j\geq 0} V_j \varepsilon^j = \sum_{k\geq 0} L_k \varepsilon^k \sum_{j\geq 0} V_j \varepsilon^j = \sum_{n\geq 0} \varepsilon^n \sum_{j=0}^{n} V_{n-j} L_j.$$

Therefore, V_j's are determined by (3.46). □

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